AS FUNCTIONS MATHS P 1

COMPILED BY : MUSTAFA ASIF

1 Pure Mathematics 1

1.2 Functions

Candidates should be able to:	Notes and examples
• understand the terms function, domain, range, one-one function, inverse function and composition of functions	
• identify the range of a given function in simple cases, and find the composition of two given functions	e.g. range of $f : x \mapsto \frac{1}{x}$ for $x \geqslant 1$ and range of $g : x \mapsto x^2 + 1$ for $x \in \mathbb{R}$. Including the condition that a composite function gf can only be formed when the range of f is within the domain of g.
• determine whether or not a given function is one-one, and find the inverse of a one-one function in simple cases	e.g. finding the inverse of $h : x \mapsto (2x+3)^2 - 4$ for $x < -\frac{3}{2}$.
• illustrate in graphical terms the relation between a one-one function and its inverse	Sketches should include an indication of the mirror line $y = x$.
• understand and use the transformations of the graph of $y = f(x)$ given by $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$, $y = f(ax)$ and simple combinations of these.	Including use of the terms 'translation', 'reflection' and 'stretch' in describing transformations. Questions may involve algebraic or trigonometric functions, or other graphs with given features.

Videos for understanding:

https://www.youtube.com/watch?v=abB9rWbOdUw

https://www.youtube.com/watch?v=BBh7rnWuo3E&list=PLjK050gbQMRIgGtgRPePpD5f11DvGrTYu&index=2

https://www.youtube.com/watch?v=KyOQhC8ctxc

A **mapping** looks at the relationship between two sets of numbers.

Consider the mapping $\times 2 + 1$

We can input a set of numbers, for example $\{-2, -1, 1, 3, 6\}$.

> The input set is called the **domain**.

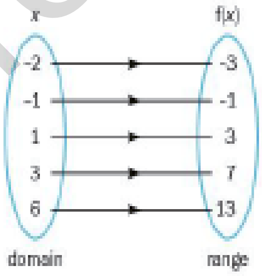
The output set can be obtained by applying the operation to each number in the input set,
e.g. input = -2, output = $-2 \times 2 + 1 = -3$.

> The output set is called the **range**.

The output set for a domain of $\{-2, -1, 1, 3, 6\}$ is $\{-3, -1, 3, 7, 13\}$.

> We say that each number is a **member** ($\in$) of that set. $x \in \mathbb{R}$ means that x is a member of the set of all the real numbers.

> A **function** is defined as a mapping where every element of the domain (x-values) is mapped onto exactly one element of the range (y-values). The diagram on the right shows an example of a function.

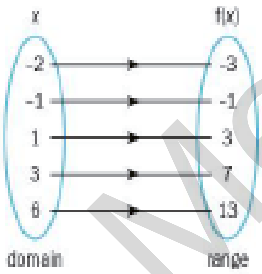


We can write this function in two different ways:

$$f(x) = 2x + 1 \quad \text{or} \quad f: x \mapsto 2x + 1$$

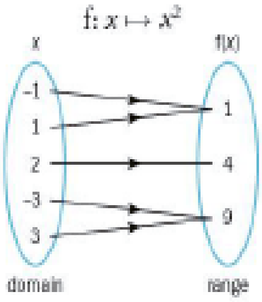
We say: f maps x onto $2x + 1$.

$f: x \mapsto 2x + 1$



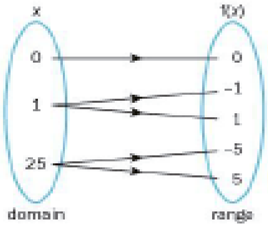
> If every element in the domain is mapped onto exactly one element in the range and every element in the range is mapped onto exactly one element in the domain, we say the function is a **one-to-one** function.

$f: x \mapsto x^2$



> If every element in the domain is mapped onto exactly one element in the range, but some elements in the range arise from more than one element in the domain, we say the function is a **many-to-one** function.

$f: x \mapsto \pm\sqrt{x}, x \geq 0$



This is **not** a function, because it is one-to-many. Some elements in the domain are mapped onto more than one element in the range.

2.2 Composite functions

When you combine two or more functions you get a composite function.

Consider the functions $f(x) = x^2 + 1, x \in \mathbb{R}$ and $g(x) = x - 1, x \in \mathbb{R}$ and the subset of numbers {1, 2, 3, 4, 5}.

If we apply function f to this set of numbers we get {2, 5, 10, 17, 26}.
If we apply function g to this new set of numbers we get {1, 4, 9, 16, 25}.

If we apply f first and then g we write this combined function as gf.

The composite function **gf(*x*)** means apply f first followed by g.

Consider the same set of numbers {1, 2, 3, 4, 5} and the functions $f(x) = x^2 + 1$ and $g(x) = x - 1$.

If we apply function g to this set of numbers we get {0, 1, 2, 3, 4}.
If we apply function f to this new set of numbers we get {1, 2, 5, 10, 17}.

If we apply g first and then f, we write this combined function as fg.

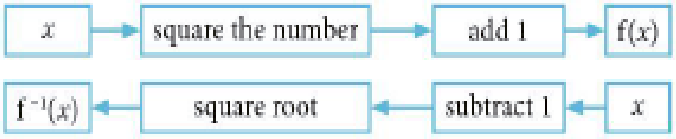
The composite function **fg(*x*)** means apply g first followed by f.

Note: fg will only exist if the range of g is contained within the domain of f. In general, $fg(x) \neq gf(x)$

2.3 Inverse functions

Consider the function $f(x) = x^2 + 1$ with domain {1, 2, 3, 4, 5}.
The range of this set of numbers is {2, 5, 10, 17, 26}.

The **inverse function** provides structured evidence that leads to a given result of f. The inverse function is written as $f^{-1}(x)$ and maps the range {2, 5, 10, 17, 26} back onto the domain {1, 2, 3, 4, 5}.

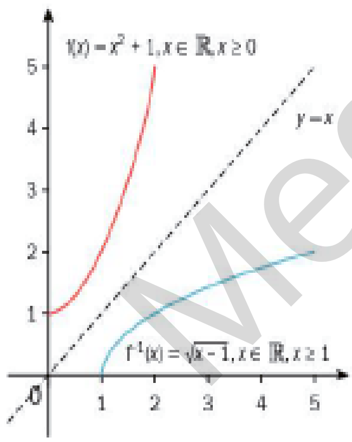


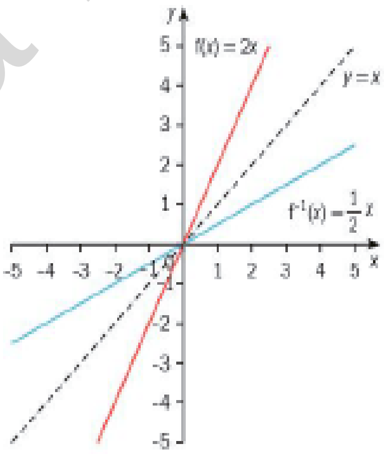
When $f(x) = x^2 + 1$, with domain {1, 2, 3, 4, 5},
$f^{-1}(x) = \sqrt{x-1}$
e.g. when $f(x) = 2$, $x = \sqrt{2-1} = 1$
e.g. when $f(x) = 26$, $x = \sqrt{26-1} = 5$

We can draw the graph of any inverse function $f^{-1}(x)$ by reflecting $f(x)$ in the line $y = x$.

Note:

- We do not write $f^{-1}(x) = \pm\sqrt{x-1}$ as all the numbers in the domain are positive.
- $f^{-1}(x) = \pm\sqrt{x-1}$ would not be a function as for every value of $x \geq 1$ we would get more than one value of $f^{-1}(x)$.
- We often have to restrict the domain so that the function is a one-to-one.
- We can only find the inverse function $f^{-1}(x)$ if $f(x)$ is a one-to-one function.





Here are some steps we will use to find the inverse function $f^{-1}(x)$ given $f(x)$.

1. Let $f(x) = y$.
2. Change x to y and y to x.
3. Rearrange to get y in terms of x.
4. Write in the correct form.

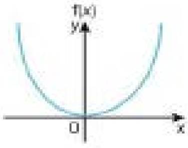
AS FUNCTIONS MATHS P1

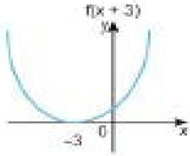
COMPILED BY : MUSTAFA ASIF

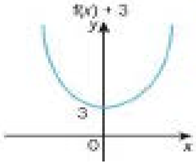
2.4 Transformations: translations

You can transform the graph of a function by moving it horizontally or vertically.
This **transformation** is called a **translation**.

> $f(x + a)$ is a horizontal translation of $-a$.
> $f(x) + a$ is a vertical translation of a.





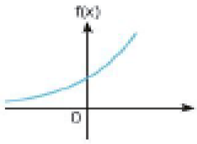


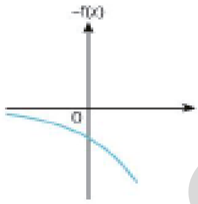
Note: We can write $y = f(x)$, $y = f(x + 3)$ and $y = f(x) + 3$ for these graphs.

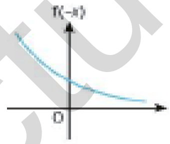
2.5 Transformations: reflections

You can transform the graph of a function by **reflecting** the graph in one of the axes.

> $-f(x)$ is a reflection in the x-axis.
> $f(-x)$ is a reflection in the y-axis.





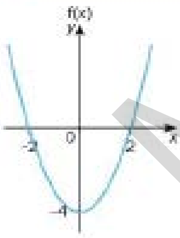


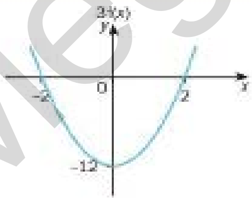
2.6 Transformations: stretches

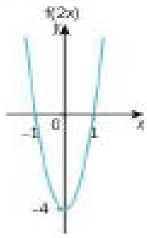
You can transform the graph of a function by **stretching** (or compressing) the graph horizontally or vertically.

> $af(x)$ is a stretch with factor a in the y-direction.
> $f(ax)$ is a stretch with factor $\frac{1}{a}$ in the x-direction.

Note: $af(x)$ means **multiply** all the y-values by a while the x-values stay the same.
$f(ax)$ means **divide** all the x-values by a while the y-values stay the same.







Chapter summary

Domain and range

- If a function maps x onto $f(x)$, then the values of x is the **domain** of the function and the corresponding values of $f(x)$ is the **range** of the function.

Functions

- A **function** is defined as a mapping where every element of the domain (set x) is mapped onto exactly one element of the range (set y).
- If every element in the domain is mapped onto exactly one element in the range we say the function is a **one-to-one** function.
- If every element in the domain is mapped onto exactly one element in the range, but some elements in the range arise from more than one element in the domain we say the function is a **many-to-one** function.
- We say that each member of a set is an **element** of that set.
- $x \in \mathbb{R}$ means that x is a member of all the real numbers.

Composite functions

- The composite function **fg(*x*)** means apply g first followed by f.
- The composite function **gf(*x*)** means apply f first followed by g.

Inverse functions

- The **inverse function** of f, written as $f^{-1}(x)$, maps the range back onto the domain.
- The steps for finding the inverse function $f^{-1}(x)$ given $f(x)$ are:
  1. Let $f(x) = y$.
  2. Change x to y and y to x.
  3. Rearrange to get y in terms of x.
  4. Write in the correct form.
- The domain of f^{-1} is always the same as the range of f.
- The range of f^{-1} is always the same as the domain of f.
- $ff^{-1}(x)$ and $f^{-1}f(x)$ are always the same as x.
- The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ in the line $y = x$.
- If $f(x) = f^{-1}(x)$, then we say $f(x)$ is a **self-inverse** function. If f is self-inverse, then $ff(x) = x$.

Transformations

- We can transform the graph of a function by moving it horizontally or vertically. This transformation is called a **translation**.
- $f(x + a)$ is a horizontal translation of $-a$.
- $f(x) + a$ is a vertical translation of a.

1 Functions f and g are defined by

$$f : x \mapsto 3x + 2, \quad x \in \mathbb{R},$$
$$g : x \mapsto 4x - 12, \quad x \in \mathbb{R}.$$

Solve the equation $f^{-1}(x) = gf(x)$. [4]

SP20/P1/2

2 The function f is defined, for $x \in \mathbb{R}$, by $f : x \mapsto x^2 + ax + b$, where a and b are constants.

(a) It is given that $a = 6$ and $b = -8$. SP20/P1/11

Find the range of f. [3]

(b) It is given instead that $a = 5$ and that the roots of the equation $f(x) = 0$ are k and $-2k$, where k is a constant.

Find the values of b and k. [3]

(c) Show that if the equation $f(x + a) = a$ has no real roots then $a^2 < 4(b - a)$. [3]

3 The function f is defined by $f(x) = -2x^2 + 12x - 3$ for $x \in \mathbb{R}$. M/J19/11/5

(i) Express $-2x^2 + 12x - 3$ in the form $-2(x + a)^2 + b$, where a and b are constants. [2]

(ii) State the greatest value of $f(x)$. [1]

The function g is defined by $g(x) = 2x + 5$ for $x \in \mathbb{R}$.

(iii) Find the values of x for which $gf(x) + 1 = 0$. [3]

4 The function f is defined by $f(x) = 2 - 3\cos x$ for $0 \leqslant x \leqslant 2\pi$.

(i) State the range of f. M/J19/11/9 [2]

(ii) Sketch the graph of $y = f(x)$. [2]

The function g is defined by $g(x) = 2 - 3\cos x$ for $0 \leqslant x \leqslant p$, where p is a constant.

(iii) State the largest value of p for which g has an inverse. [1]

(iv) For this value of p, find an expression for $g^{-1}(x)$. [2]

5 Functions f and g are defined by

M/J19/12/7

$$f : x \mapsto 3x - 2, \quad x \in \mathbb{R},$$

$$g : x \mapsto \frac{2x+3}{x-1}, \quad x \in \mathbb{R}, \ x \neq 1.$$

(i) Obtain expressions for $f^{-1}(x)$ and $g^{-1}(x)$, stating the value of x for which $g^{-1}(x)$ is not defined. [4]

(ii) Solve the equation $fg(x) = \frac{7}{3}$. [3]

6 The function f is defined by $f(x) = \frac{48}{x-1}$ for $3 \leqslant x \leqslant 7$. The function g is defined by $g(x) = 2x - 4$ for $a \leqslant x \leqslant b$, where a and b are constants.

(i) Find the greatest value of a and the least value of b which will permit the formation of the composite function gf. [2]

M/J19/13/4

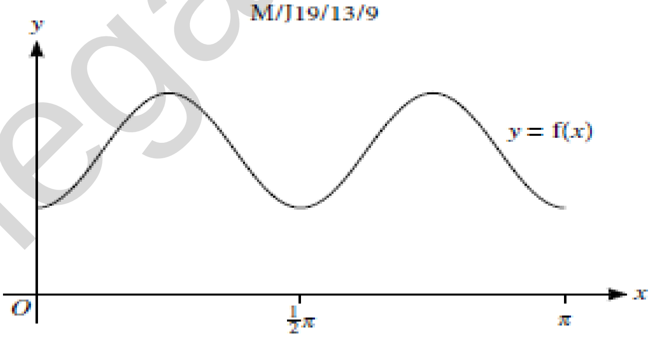
It is now given that the conditions for the formation of gf are satisfied.

(ii) Find an expression for $gf(x)$. [1]

(iii) Find an expression for $(gf)^{-1}(x)$. [2]

6

M/J19/13/9



The function $f : x \mapsto p\sin^2 2x + q$ is defined for $0 \leqslant x \leqslant \pi$, where p and q are positive constants. The diagram shows the graph of $y = f(x)$.

(i) In terms of p and q, state the range of f. [2]

(ii) State the number of solutions of the following equations.

(a) $f(x) = p + q$ [1]

(b) $f(x) = q$ [1]

(c) $f(x) = \frac{1}{2}p + q$ [1]

(iii) For the case where $p = 3$ and $q = 2$, solve the equation $f(x) = 4$, showing all necessary working. [5]

7 The function f is defined by $f(x) = x^2 - 4x + 8$ for $x \in \mathbb{R}$.

M/J19/13/1

(i) Express $x^2 - 4x + 8$ in the form $(x - a)^2 + b$. [2]

(ii) Hence find the set of values of x for which $f(x) < 9$, giving your answer in exact form. [3]

8 The function f is defined by $f(x) = \dfrac{48}{x - 1}$ for $3 \leqslant x \leqslant 7$. The function g is defined by $g(x) = 2x - 4$ for $a \leqslant x \leqslant b$, where a and b are constants.

(i) Find the greatest value of a and the least value of b which will permit the formation of the composite function gf. [2]

M/J19/13/4

It is now given that the conditions for the formation of gf are satisfied.

(ii) Find an expression for $gf(x)$. [1]

(iii) Find an expression for $(gf)^{-1}(x)$. [2]

9 (i) Express $x^2 - 4x + 7$ in the form $(x + a)^2 + b$. M19/12/8 [2]

The function f is defined by $f(x) = x^2 - 4x + 7$ for $x < k$, where k is a constant.

(ii) State the largest value of k for which f is a decreasing function. [1]

The value of k is now given to be 1.

(iii) Find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [3]

10 (a) The one-one function f is defined by $f(x) = (x-3)^2 - 1$ for $x < a$, where a is a constant.

(i) State the greatest possible value of a. [1]

(ii) It is given that a takes this greatest possible value. State the range of f and find an expression for $f^{-1}(x)$. [3]

(b) The function g is defined by $g(x) = (x-3)^2$ for $x \geqslant 0$.

(i) Show that $gg(2x)$ can be expressed in the form $(2x-3)^4 + b(2x-3)^2 + c$, where b and c are constants to be found. [2]

(ii) Hence expand $gg(2x)$ completely, simplifying your answer. [4]

11 The function f is defined by $f : x \mapsto 2x^2 - 12x + 7$ for $x \in \mathbb{R}$. O/N18/12/9

(i) Express $2x^2 - 12x + 7$ in the form $2(x+a)^2 + b$, where a and b are constants. [2]

(ii) State the range of f. [1]

The function g is defined by $g : x \mapsto 2x^2 - 12x + 7$ for $x \leqslant k$.

(iii) State the largest value of k for which g has an inverse. [1]

(iv) Given that g has an inverse, find an expression for $g^{-1}(x)$. [3]

12 Functions f and g are defined by

$$f : x \mapsto 2 - 3\cos x \quad \text{for } 0 \leqslant x \leqslant 2\pi,$$
$$g : x \mapsto \tfrac{1}{2}x \quad \text{for } 0 \leqslant x \leqslant 2\pi.$$

(i) Solve the equation $fg(x) = 1$. O/N18/12/4 [3]

(ii) Sketch the graph of $y = f(x)$. [3]

13 (i) Express $2x^2 - 12x + 11$ in the form $a(x+b)^2 + c$, where a, b and c are constants. [3]

o/n18/13/11

The function f is defined by $f(x) = 2x^2 - 12x + 11$ for $x \leqslant k$.

(ii) State the largest value of the constant k for which f is a one-one function. [1]

(iii) For this value of k find an expression for $f^{-1}(x)$ and state the domain of f^{-1}. [4]

The function g is defined by $g(x) = x + 3$ for $x \leqslant p$.

(iv) With k now taking the value 1, find the largest value of the constant p which allows the composite function fg to be formed, and find an expression for $fg(x)$ whenever this composite function exists. [3]

AS FUNCTIONS MATHS P 1

COMPILED BY : MUSTAFA ASIF

14 The function f is defined by $f(x) = x^3 + 2x^2 - 4x + 7$ for $x \geqslant -2$. Determine, showing all necessary working, whether f is an increasing function, a decreasing function or neither. [4]

O/N18/13/2

---

15 Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto \tfrac{1}{2}x - 2,$$

M/J18/11/9

$$g : x \mapsto 4 + x - \tfrac{1}{2}x^2.$$

(i) Find the points of intersection of the graphs of $y = f(x)$ and $y = g(x)$. [3]

(ii) Find the set of values of x for which $f(x) > g(x)$. [2]

(iii) Find an expression for $fg(x)$ and deduce the range of fg. [4]

The function h is defined by $h : x \mapsto 4 + x - \tfrac{1}{2}x^2$ for $x \geqslant k$.

(iv) Find the smallest value of k for which h has an inverse. [2]

---

16 The function f is such that $f(x) = a + b\cos x$ for $0 \leqslant x \leqslant 2\pi$. It is given that $f(\tfrac{1}{3}\pi) = 5$ and $f(\pi) = 11$.

(i) Find the values of the constants a and b. M/J18/12/4 [3]

(ii) Find the set of values of k for which the equation $f(x) = k$ has no solution. [3]

17 The function f is defined by $f : x \mapsto 7 - 2x^2 - 12x$ for $x \in \mathbb{R}$.

(i) Express $7 - 2x^2 - 12x$ in the form $a - 2(x + b)^2$, where a and b are constants. [2]

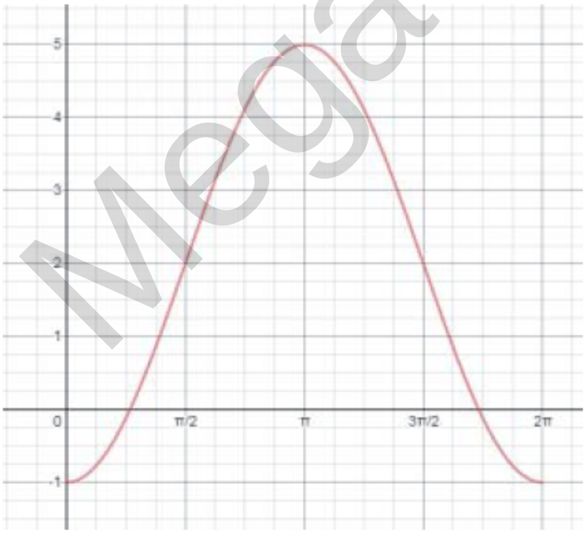
Marking Scheme

Q	Answer		Marks	Guidance
1	$f^{-1}(x) = \frac{x-2}{3}$	1	B1	
	$gf(x) = 4(3x+2) - 12$	1	B1	
	Equate $f^{-1}(x)$ and $gf(x)$ expressions, $x = \frac{2}{7}$	2	M1A1	
		4		

2(a)	$x^2 + 6x - 8 = (x+3)^2 - 17$ OR $2x + 6 = 0 \to x = -3 \to y = -17$	2	B1B1	B1 for $(x+3)^2$, B1 for -17 OR B1 for $x = -3$, B1 for $y = -17$
	Range $f(x) \geqslant -17$	1	B1FT	FT; following through visible method
		3		
2(b)	$(x-k)(x+2k) = 0 \equiv x^2 + 5x + b = 0$	1	M1	Realises the link between roots and the equation
	$k = 5$	1	A1	Comparing coefficients of x
	$b = -2k^2 = -50$	1	A1	
		3		
2(c)	$(x+a)^2 + a(x+a) + b = a$	1	M1*	Replaces 'x' by '$x + a$' in 2 terms
	Uses $b^2 - 4ac$, $9a^2 - 4(2a^2 + b - a)$	1	DM1	Any use of discriminant
	$a^2 < 4(b-a)$	1	A1	AG
		3		

3(i)	$-2(x-3)^2+15$ $(a=-3, b=15)$	B1 B1	Or seen as $a=-3$, $b=15$ B1 for each value
		2	
3(ii)	$(f(x) \leqslant)$ 15	B1	FT for $(\leqslant)$ their "b" Don't accept (3,15) alone
		1	
3(iii)	$gf(x)=2(-2x^2+12x-3)+5=-4x^2+24x-6+5$	B1	
	$gf(x)+1=0 \rightarrow -4x^2+24x=0$	M1	
	$x=0$ or 6	A1	Forms and attempts to solve a quadratic Both answers given.
		3	

4(i)	$-1 \leqslant f(x) \leqslant 5$ or $[-1, 5]$ (may use y or f instead of f(x))	B1 B1	$-1 < f(x) \leqslant 5$ or $-1 \leqslant x \leqslant 5$ or $(-1,5)$ or $[5,-1]$ B1 only
		2	

Question	Answer	Marks	Guidance
4(ii)		*B1	Start and end at –ve y, symmetrical, centre +ve.
	$g(x) = 2-3\cos x$ for $0 \leqslant x \leqslant p$	DB1	Shape all ok. Curves not lines. One cycle $[0,2\pi]$ Flattens at each end.
		2	

4(iii)	(greatest value of p =) π	B1	
		1	
4(iv)	$x = 2 - 3\cos x \rightarrow \cos x = ⅓(2 - x)$	M1	Attempt at cos$x$ the subject. Use of $\cos^{-1}$
	$g^{-1}(x) = \cos^{-1}\frac{2-x}{3}$ (may use 'y =')	A1	Must be a function of x,
		2	

5(i)	$(f^{-1}(x)) = \frac{x+2}{3}$ oe	B1	
	$y = \frac{2x+3}{x-1} \rightarrow (x-1)y = 2x+3 \rightarrow x(y-2) = y+3$	M1	Correct method to obtain $x =$, (or $y =$, if interchanged) but condone $+/-$ sign errors
	$(g^{-1}(x)$ or $y) = \frac{x+3}{x-2}$ oe $\left(eg \frac{5}{x-2}+1\right)$	A1	Must be in terms of x
	$x \neq 2$ only	B1	FT for value of x from their denominator = 0
		4	
5(ii)	$(fg(x) =) \frac{3(2x+3)}{x-1} - 2\ (= \frac{7}{3})$	B1	
	$18x + 27 = 13x - 13$ or $3(4x + 11) = 7(x - 1)$ $(5x = -40)$	M1	Correct method from their $fg = \frac{7}{3}$ leading to a linear equation and collect like terms. Condone omission of $2(x-1)$.
	Alternative method for question 7(ii)		
	$(f^{-1}(\frac{7}{3})) = \frac{13}{9}$	B1	
	$\frac{2x+3}{x-1} = \frac{13}{9} \rightarrow 9(2x+3) = 13(x-1)\ (\rightarrow 5x = -40)$	M1	Correct method from $g(x) =$ their $\frac{13}{9}$ leading to linear equation and collect like terms.
	$x = -8$	A1	
		3	

6(i)	$q \leqslant f(x) \leqslant p+q$	B1B1	B1 each inequality – allow two separate statements Accept $<$, $(q, p+q)$, $[q, p+q]$ Condone y or x or f in place of f(x)
		2	

6(ii)	**(a)** 2	**B1**	Allow $\frac{\pi}{4}, \frac{3\pi}{4}$
	(b) 3	**B1**	Allow $0, \frac{\pi}{2}, \pi$
	(c) 4	**B1**	Allow $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$
		3	
6(iii)	$3\sin^2 2x + 2 = 4 \rightarrow \sin^2 2x = \frac{2}{3}$ soi	**M1**	
	$\text{Sin}2x = (\pm)0.816(5)$. Allow $\sin 2x = (\pm)\sqrt{\frac{2}{3}}$ or $2x = \sin^{-1}(\pm)\sqrt{\frac{2}{3}}$	**A1**	OR Implied by at least one correct value for x. Allow $\sin^{-1}$ form
	$(2x =)$ **at least two** of 0.955(3), 2.18(6), 4.09(7), 5.32(8)	**A1**	Can be implied by corresponding values of x below Allow for at least two of 0.304π, 0.696π, $1.30(4)\pi$, $1.69(6)\pi$ **OR** at least two of 54.7(4)°, 125.2(6)°, 234.7(4)°, 305.2(6)°
	$(x =)$ 0.478, 1.09, 2.05, 2.66.	**A1A1**	Allow 0.152π, 0.348π, 0.652π, 0.848π SC A1 for 2 or 3 correct. SC A1 for all of 27.4°, 62.6°, 117.4°, 152.6° $\text{Sin}2x = \pm\frac{2}{3} \rightarrow x = 0.365, 1.21, 1.94, 2.78$ scores SC M1A0A0A1
		5	

7(i)	$\left[(x-2)^2\right]$ $[+4]$	**B1 DB1**	2nd B1 dependent on 2 inside bracket
		2	
7(ii)	$(x-2)^2 < 5 \rightarrow -\sqrt{5} < x-2$ and/or $x-2 < \sqrt{5}$	**M1**	Allow e.g. $x-2 < \pm\sqrt{5}$, $x-2 = \pm\sqrt{5}$ and decimal equivalents for 5 For M1, ft from *their*(i). Also allow √13 instead of √5 for clear sli
	$2-\sqrt{5} < x < 2+\sqrt{5}$	**A1A1**	A1 for each inequality – allow two separate statements but there m st be 2 inequalities for x. Non-hence methods, if completely correct, score SC 1/3. Condone $\leqslant$
		[3]	

AS FUNCTIONS MATHS P 1

COMPILED BY : MUSTAFA ASIF

8(i)	Max(a) is 8	B1	Allow $a = 8$ or $a \leqslant 8$
	Min(b) is 24	B1	Allow $b = 24$ or $b \geqslant 24$
		2	SCB1 for 8 and 24 seen
8(ii)	$\text{gf}(x) = \frac{96}{x-1} - 4$ or $\text{gf}(x) = \frac{100-4x}{x-1}$	B1	$2\left(\frac{48}{x-1}\right) - 4$ is insufficient Apply ISW
		1	
8(iii)	$y = \frac{96}{x-1} - 4 \rightarrow y + 4 = \frac{96}{x-1} \rightarrow x - 1 = \frac{96}{y+4}$	M1	FT from *their*(ii) provided (ii) involves algebraic fraction. Allow sign errors
	$(\text{gf})^{-1}(x) = \frac{96}{x+4} + 1$	A1	OR $\frac{100+x}{x+4}$. Must be a function of x. Apply ISW
		2	

9(i)	$\left[(x-2)^2\right] + [3]$	B1 DB1	2nd B1 dependent on ±2 in 1st bracket
		2	
9(ii)	Largest k is 2 Accept $k \leqslant 2$	B1	Must be in terms of k
		1	
9(iii)	$y = (x-2)^2 + 3 \Rightarrow x - 2 = (\pm)\sqrt{y-3}$	M1	
	$\Rightarrow \text{f}^{-1}(x) = 2 - \sqrt{x-3}$ for $x > 4$	A1B1	
		3	

9(iv)	$\text{gf}(x) = \frac{2}{x^2 - 4x + 7 - 1} = \frac{2}{(x-2)^2 + 2}$	B1	Either form
	Since $\text{f}(x) > 4 \Rightarrow \text{gf}(x) < 2/3$ (or since $x < 1$etc)	M1A1	2/3 in answer implies M1 www
	range of gf(x) is $0 < \text{gf}(x) (< 2/3)$	B1	Accept $0 < y < 2/3$, (0, 2/3) but $0 < x < 2/3$ is SCM1A1B0
		4	

10(a)(i)	[Greatest value of a is] 3	B1	Must be in terms of a. Allow $a<3$. Allow $a \leqslant 3$
		1	
10(a)(ii)	Range is $y>-1$	B1	Ft on *their* a. Accept any equivalent notation
	$y=(x-3)^2-1 \to (x-3)^2=1+y \to x=3(\pm)\sqrt{1+y}$	M1	Order of operations correct. Allow sign errors
	$f^{-1}(x)=3-\sqrt{1+x}$ cao	A1	
		3	
10(b)(i)	gg(2x) $=\left[(2x-3)^2-3\right]^2$	B1	
	$(2x-3)^4-6(2x-3)^2+9$	B1	
		2	
10(b)(ii)	$\left[16x^4-96x^3+216x^2-216x+81\right]+\left[(-24x^2+72x-54)+9\right]$ $16x^4-96x^3+192x^2-144x+36$	B4,3,2,1,0	
		4	

11(i)	$2x^2-12x+7=2(x-3)^2-11$	B1 B1	Mark full expression if present: B1 for $2(x-3)^2$ and B1 for -11. If no clear expression award $a=-3$ and $b=-11$.
		2	
11(ii)	Range (of f or y) $\geqslant$ 'their -11'	B1FT	FT for their 'b' or start again. Condone >. Do **NOT** accept x > or $\geqslant$
		1	
11(iii)	(k =) –"their a" also allow x or $k \leqslant 3$	B1FT	FT for their "a" or start again using $\frac{dy}{dx}=0$. Do **NOT accept** $x = 3$.
		1	
11(iv)	$y=2(x-3)^2-11 \to y+11=2(x-3)^2$ $\frac{y+11}{2}=(x-3)^2$	*M1	Isolating their $(x-3)^2$, condone -11.
	$x=3+\sqrt{\left(\frac{y+11}{2}\right)}$ or $3-\sqrt{\left(\frac{y+11}{2}\right)}$	DM1	Other operations in correct order, allow ± at this stage. Condone -3.
	($g^{-1}(x)$ or y) $=3-\sqrt{\left(\frac{x+11}{2}\right)}$	A1	needs '–'. x and y could be interchanged at the start.
		3	

12(i)	$fg(x) = 2 - 3\cos(\frac{1}{2}x)$	B1	Correct fg
	$2 - 3\cos(\frac{1}{2}x) = 1 \rightarrow \cos(\frac{1}{2}x) = \frac{1}{3} \rightarrow \left(\frac{1}{2}x\right) = \cos^{-1}\left(their\frac{1}{3}\right)$	M1	M1 for correct order of operations to solve their $fg(x) = 1$ as far as using inverse cos expect 1.23, (or 70.5°) condone $x =$.
	$x =$ **2.46 awrt or** $\frac{4.7\pi}{6}$ $(0.784\pi$ **awrt**$)$	A1	One solution only in the given range, ignore answers outside the range. Answer in degrees A0.
			Alternative: Solve $f(y) = 1 \rightarrow y = 1.23 \rightarrow \frac{1}{2}x = 1.23$ **B1M1** $\rightarrow x = 2.46$ **A1**
		3	

12(ii)		B1	One cycle of ± cos curve, evidence of turning at the ends not required at this stage. Can be a poor curve but not an inverted "V". If horizontal axis is not labelled mark everything to the right of the vertical axis. If axis is clearly labelled mark 0 → 2π.
		B1	Start and finish at roughly the same negative y value. Significantly more above the x axis than below or correct range implied by labels .
		B1	Fully correct. Curves not lines. Must be a reasonable curve clearly turning at both ends. Labels not required but must be appropriate if present.
		3	

13(i)	[2] $\left[(x-3)^2\right]$ [−7]	B1B1B1	
		3	
13(ii)	Largest value of k is 3. Allow (k =) 3.	B1	Allow $k \leqslant 3$ but not $x \leqslant 3$ as final answer.
		1	

13(iii)	$y = 2(x-3)^2 - 7 \to (x-3)^2 = ½(y+7)$ or with x/y transposed	M1	Ft *their* a, b, c. Order of operations correct. Allow sign errors
	$x = 3 \pm \sqrt{½(y+7)}$ Allow 3 + √ or 3 −√ or with x/y transposed	DM1	Ft *their* a, b, c. Order of operations correct. Allow sign errors
	$f^{-1}(x) = 3 - \sqrt{½(x+7)}$	A1	
	(Domain is x) $\geqslant$ *their* − 7	B1FT	Allow other forms for interval but if variable appears must be x
		4	
13(iv)	$x + 3 \leqslant 1$. Allow x + 3 = 1	M1	Allow $x + 3 \leqslant k$
	largest p is − 2 . Allow (p =) −2	A1	Allow $p \leqslant -2$ but not $x \leqslant -2$ as final answer.
	$fg(x) = f(x+3) = 2x^2 - 7$ cao	B1	
		3	

14	$f'(x) = 3x^2 + 4x - 4$	B1	
	Factors or crit. values or sub any 2 values $(x \neq -2)$ into $f'(x)$ soi	M1	Expect $(x+2)(3x-2)$ or −2, ⅔ or any 2 subs (excluding x = −2).
	For $-2 < x < ⅔$, $f'(x) < 0$; for $x > ⅔$, $f'(x) > 0$ soi Allow $\leqslant, \geqslant$	M1	Or at least 1 specific value ($\neq -2$) in each interval giving opp signs Or $f'(⅔)$=0 and $f''(⅔) \neq 0$ (i.e. gradient changes sign at x = ⅔)
	Neither www	A1	Must have 'Neither'
	ALT 1 At least 3 values of f(x)	M1	e.g. f(0) = 7, f(1) = 6, f(2) = 15
	At least 3 correct values of f(x)	A1	
	At least 3 correct values of f(x) spanning x = ⅔	A1	
	Shows a decreasing and then increasing pattern. Neither www	A1	Or similar wording. Must have 'Neither'
	ALT 2 $f'(x) = 3x^2 + 4x - 4 = 3(x+⅔)^2 - 16/3$	B1B1	Do not condone sign errors
	$f'(x) \geqslant -\frac{16}{3}$	M1	
	$f'(x) < 0$ for some values and > 0 for other values. Neither www	A1	Or similar wording. Must have 'Neither'
		4	

15	$f : x \mapsto \frac{x}{2} - 2, \quad g : x \mapsto 4 + x - \frac{x^2}{2}$		
15(i)	$4 + x - \frac{x^2}{2} = \frac{x}{2} - 2 \rightarrow x^2 - x - 12 = 0$	M1	Equates and forms 3 term quadratic
	$\rightarrow$ (4, 0) and (−3, −3.5) Trial and improvement, B3 all correct or B0	A1 A1	A1 For both x values or a correct pair. A1 all.
		3	
15(ii)	$f(x) > g(x)$ for $x > 4, x < -3$	B1, B1	B1 for each part. Loses a mark for $\leqslant$ or $\geqslant$.
		2	
15(iii)	$fg(x) = 2 + \frac{x}{2} - \frac{x^2}{4} - 2 \left(= \frac{x}{2} - \frac{x^2}{4}\right)$	B1	CAO, any correct form
	i.e. $-\frac{1}{4}((x-1)^2 - 1)$ or $\frac{dy}{dx} = \frac{1}{2} - \frac{2x}{4} = 0 \rightarrow x = 1$	M1 A1	Completes the square or uses calculus. First A1 is for $x = 1$ or completed square form
	$\rightarrow y = \frac{1}{4} \rightarrow$ Range of fg $\leqslant \frac{1}{4}$,	A1	CAO, OE e.g. $y \leqslant \frac{1}{4}$, $[-\infty, \frac{1}{4})$ etc.
		4	
15(iv)	Calculus or completing square on 'h' $\rightarrow x = 1$	M1	May use a sketch or $-\frac{b}{2a}$
	$k = 1$ (accept $k \geqslant 1$)	A1	Complete method. CAO
		2	

16(i)	$a + ½b = 5$	B1	Alternatively these marks can be awarded when ½ and −1 appear after a or b has been eliminated.
	$a - b = 11$	B1	
	$\rightarrow a = 7$ and $b = -4$	B1	
		[3]	
16(ii)	$a + b$ or *their* a + *their* b (3)	B1	Not enough to be seen in a table of values – must be selected. Graph from their values can get both marks. **Note: Use of $b^2 - 4ac$ scores 0/3**
	$a - b$ or *their* a – *their* b (11).	B1	
	$\rightarrow k < 3$, $k > 11$	B1	Both inequalities correct. Allow combined statement as long as correct inequalities if taken separately. Both answers correct from T & I or guesswork 3/3 otherwise 0/3
		3	

17(i)	$25 - 2(x + 3)^2$	B1 B1	Mark expression if present: B1 for 25 and B1 for $-2(x + 3)^2$. If no expression award $a = 25$ B1 and $b = 3$ B1.
		2	
17(ii)	(−3, 25)	B1FT	FT from answers to (i) or by calculus
		1	
17(iii)	$(k) = -3$ also allow x or $k \geqslant -3$	B1FT	FT from answer to (i) or (ii) NOT $x = -3$
		1	

17(iv)	**EITHER**		
	$y = 25 - 2(x+3)^2 \rightarrow 2(x+3)^2 = 25 - y$	*M1	Makes their squared term containing x the subject or equivalent with x/y interchanged first. Condone errors with +/- signs.
	$x + 3 = (\pm)\sqrt{½(25-y)}$	DM1	Divide by ±2 and then square root allow ±.
	OR		
	$y = 7 - 2x^2 - 12x \rightarrow 2x^2 + 12x + y - 7\ (= 0)$	*M1	Rearranging equation of the curve.
	$x = \frac{-12 \pm \sqrt{12^2 - 8(y-7)}}{4}$	DM1	Correct use of their 'a, b and c' in quadratic formula. Allow just + in place of ±.
	$g^{-1}(x) = \sqrt{\left(\frac{25-x}{2}\right)} - 3$ oe isw if substituting $x = -3$	A1	± gets A0. Must now be a function of x. Allow $y =$
		3	