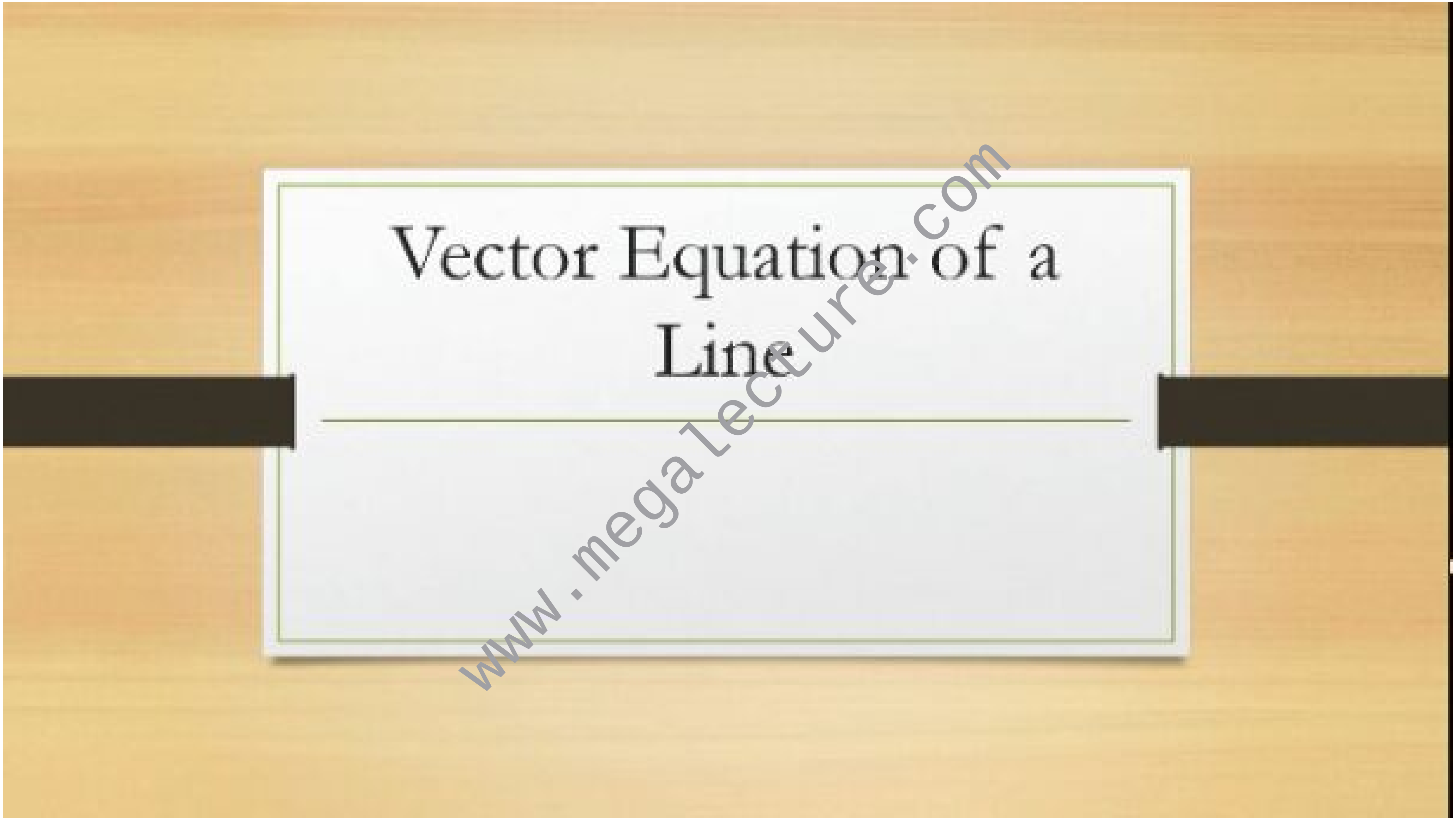
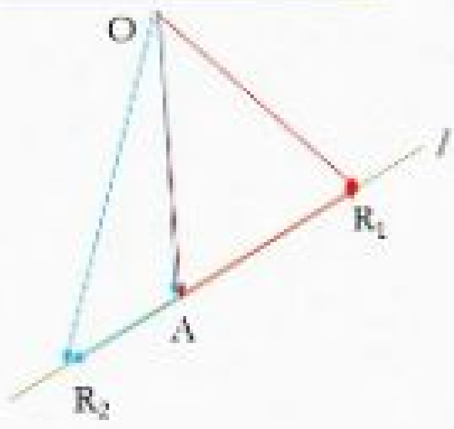

Vector Equation of a Line

MEGA LECTURE

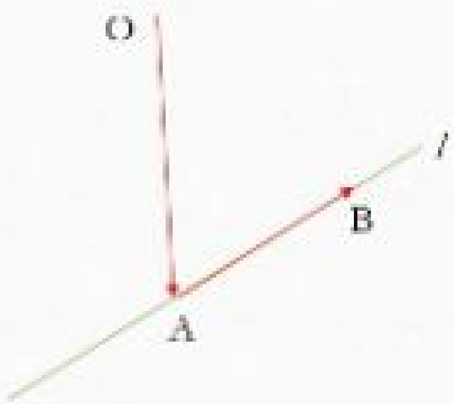
Vector Equation

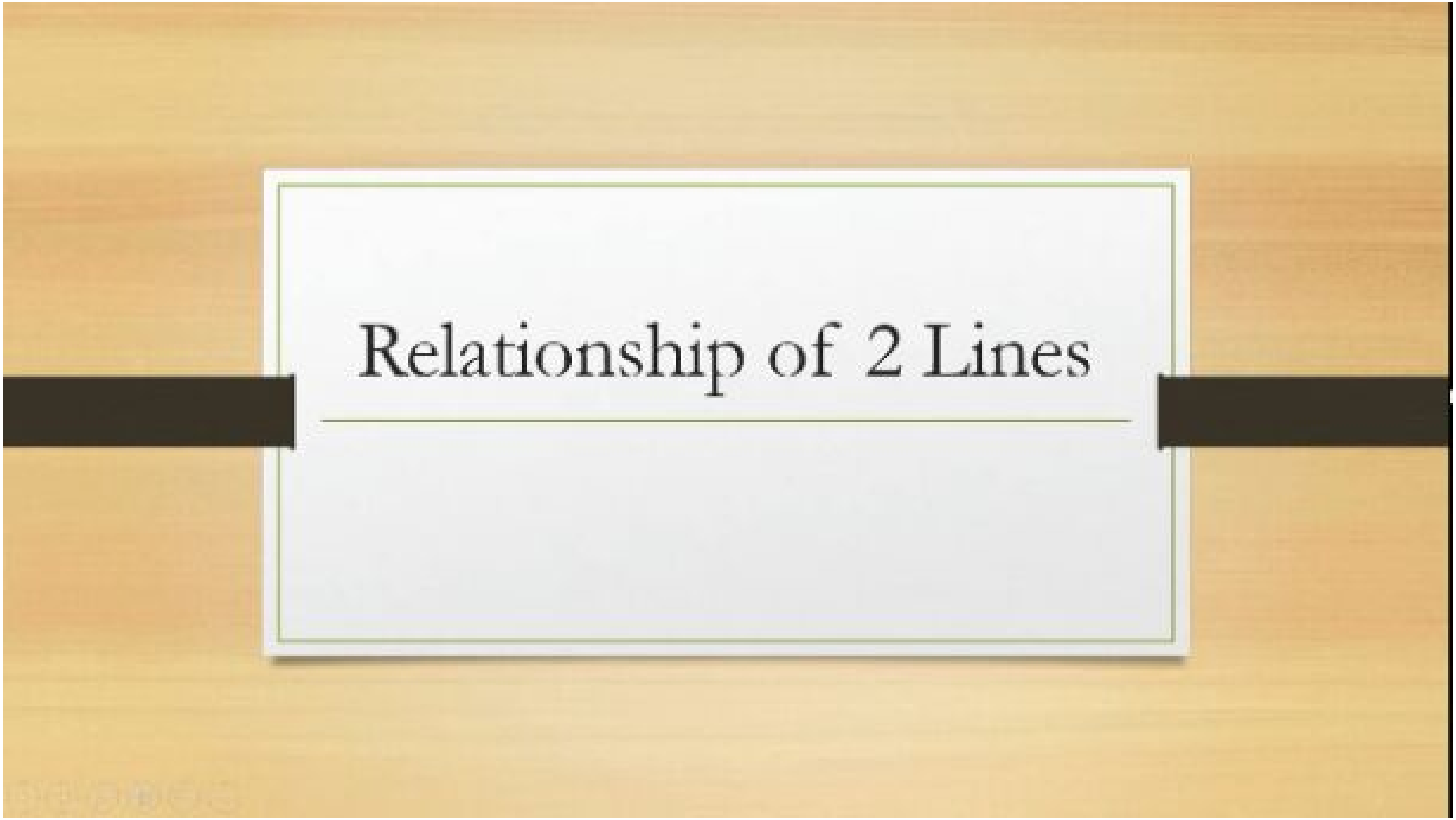
- O is the origin. A is a given point on the line.
- $\overrightarrow{OR_1} = \overrightarrow{OA} + \overrightarrow{AR_1}$
- $\overrightarrow{OR_2} = \overrightarrow{OA} + \overrightarrow{AR_2}$
- $\overrightarrow{OR_n} = \overrightarrow{OA} + \overrightarrow{AR_n}$
- $\overrightarrow{AR_1} \, // \, \overrightarrow{AR_2} \, // \cdots // \, \overrightarrow{AR_n}$
- Given the direction vector $\boldsymbol{b}$, AR_1, AR_2, … and AR_n are all multiples of $\boldsymbol{b}$.
- Given a point $\boldsymbol{a}$ and direction vector $\boldsymbol{b}$, the vector equation of a line is $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$.



Example

- Find the vector equation of a line between 2 points A(1, 2, 3) and B(6, 5, 4).
- Direction vector is $\begin{pmatrix}6\\5\\4\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}5\\3\\1\end{pmatrix}$
- So the equation is $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}5\\3\\1\end{pmatrix}$
- Or $\boldsymbol{r} = \begin{pmatrix}6\\5\\4\end{pmatrix} + t\begin{pmatrix}5\\3\\1\end{pmatrix}$
- Vector equation is NOT unique!




Relationship of 2 Lines

Parallel

- Two lines are parallel if their direction vectors are parallel.
- $\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$ and $\boldsymbol{r} = \begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix} + t\begin{pmatrix} -10 \\ -6 \\ 2 \end{pmatrix}$ are parallel.
- $\begin{pmatrix} -10 \\ -6 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}$

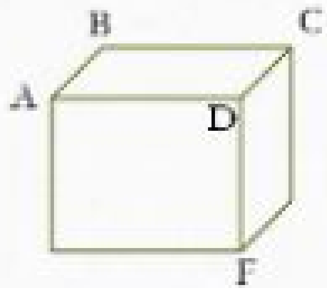
MEGA LECTURE

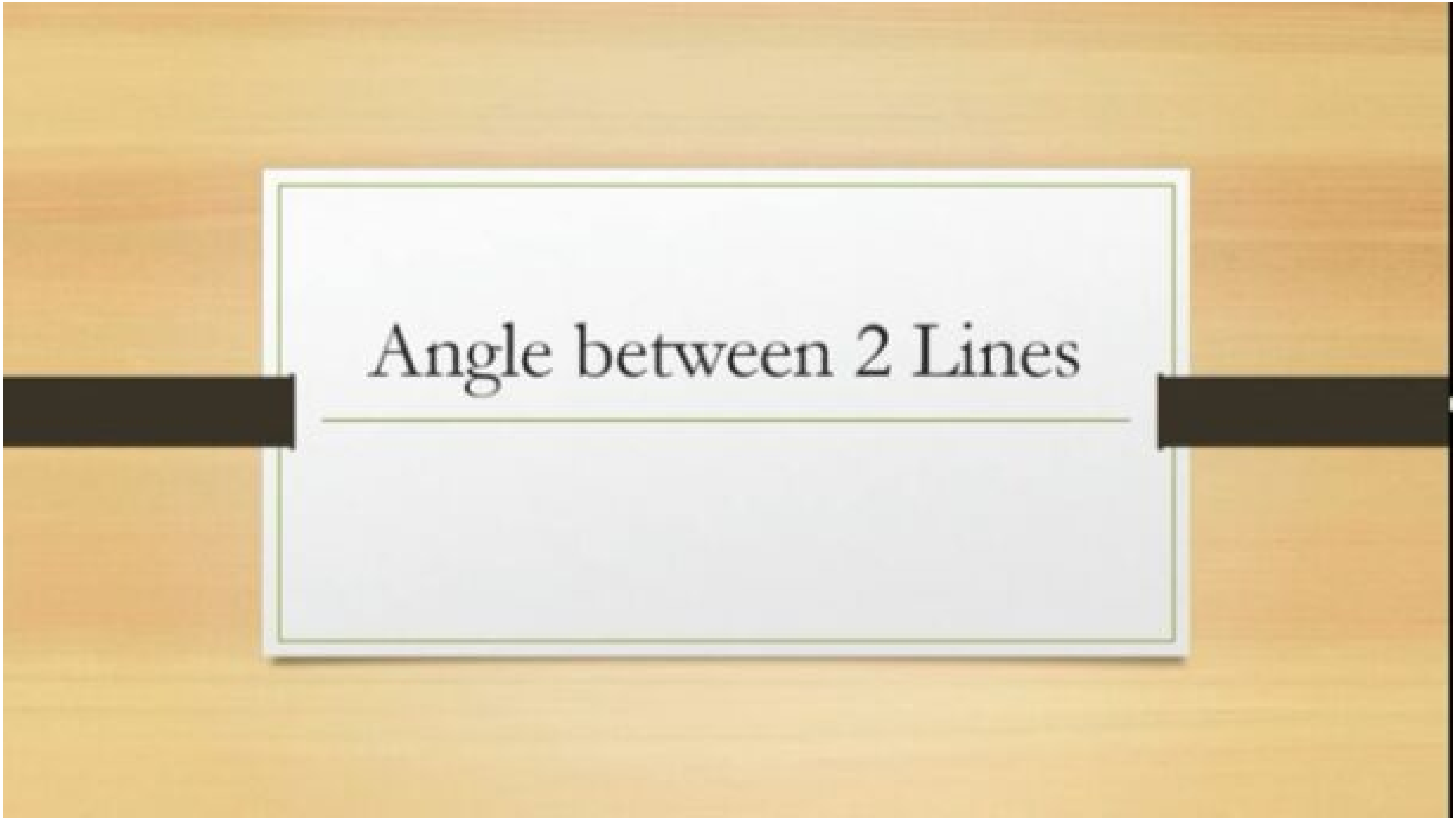
Intersect

- Find the intersection of two lines $r = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $r = \begin{pmatrix}2\\4\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\\4\end{pmatrix}$
- $r = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}1+2s\\2+3s\\3+4s\end{pmatrix}$ $\quad r = \begin{pmatrix}2\\4\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\\-4\end{pmatrix} = \begin{pmatrix}2+t\\4+2t\\-1-4t\end{pmatrix}$
- $\begin{cases}1+2s = 2+t\\2+3s = 4+2t\\3+4s = -1-4t\end{cases}$ From the first 2 equations, we get $s = 0$, $t = -1$
- Since there are 2 variables and 3 equations, we must substitute s and t into the 3rd equation.
- $3 + 4(0) = 3$, $-1-4(-1) = 3$. They are equal. So it's good.
- The intersection is $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Skew

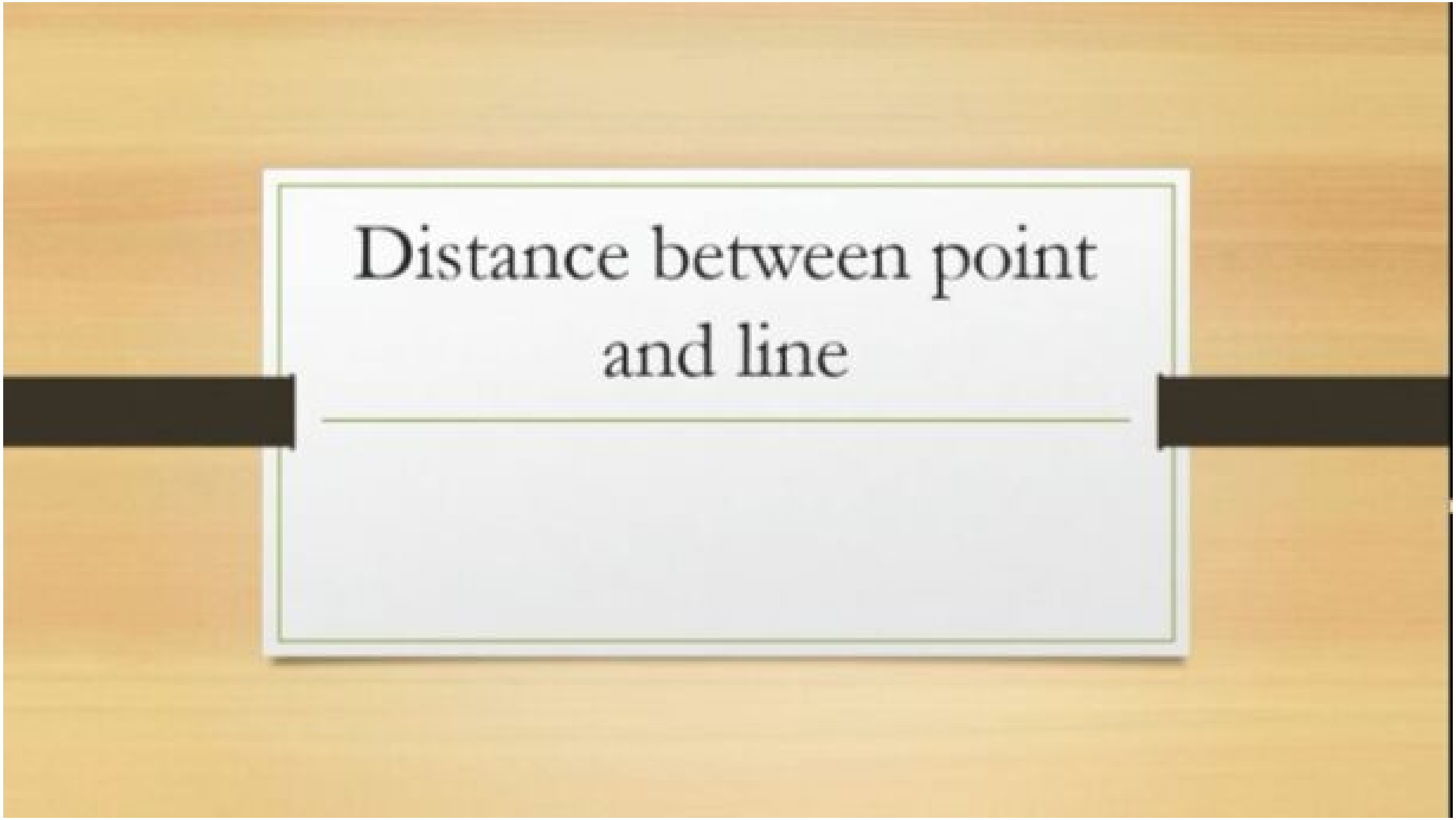
- In 3-D, skew lines don't intersect and are not parallel. AB and DF are skew.
- $r = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}2\\3\\4\end{pmatrix}$ and $r = \begin{pmatrix}2\\4\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}$.
- $r = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}2\\3\\4\end{pmatrix} = \begin{pmatrix}1+2s\\2+3s\\3+4s\end{pmatrix}$ $\quad r = \begin{pmatrix}2\\4\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2+t\\4+2t\\-1+3t\end{pmatrix}$
- $\begin{cases}1+2s=2+t\\2+3s=4+2t\\3+4s=-1+3t\end{cases}$ From the first 2 equations, we get $s=0, t=-1$
- Substitute s and t into the 3rd equation: LHS = 3 + 4(0) = 3, RHS = −1 + 3(−1) = −4. They are NOT equal!
- So these 2 lines are skew.




Angle between 2 Lines

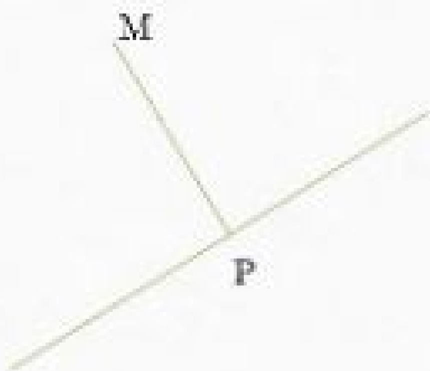
Angle between 2 Lines

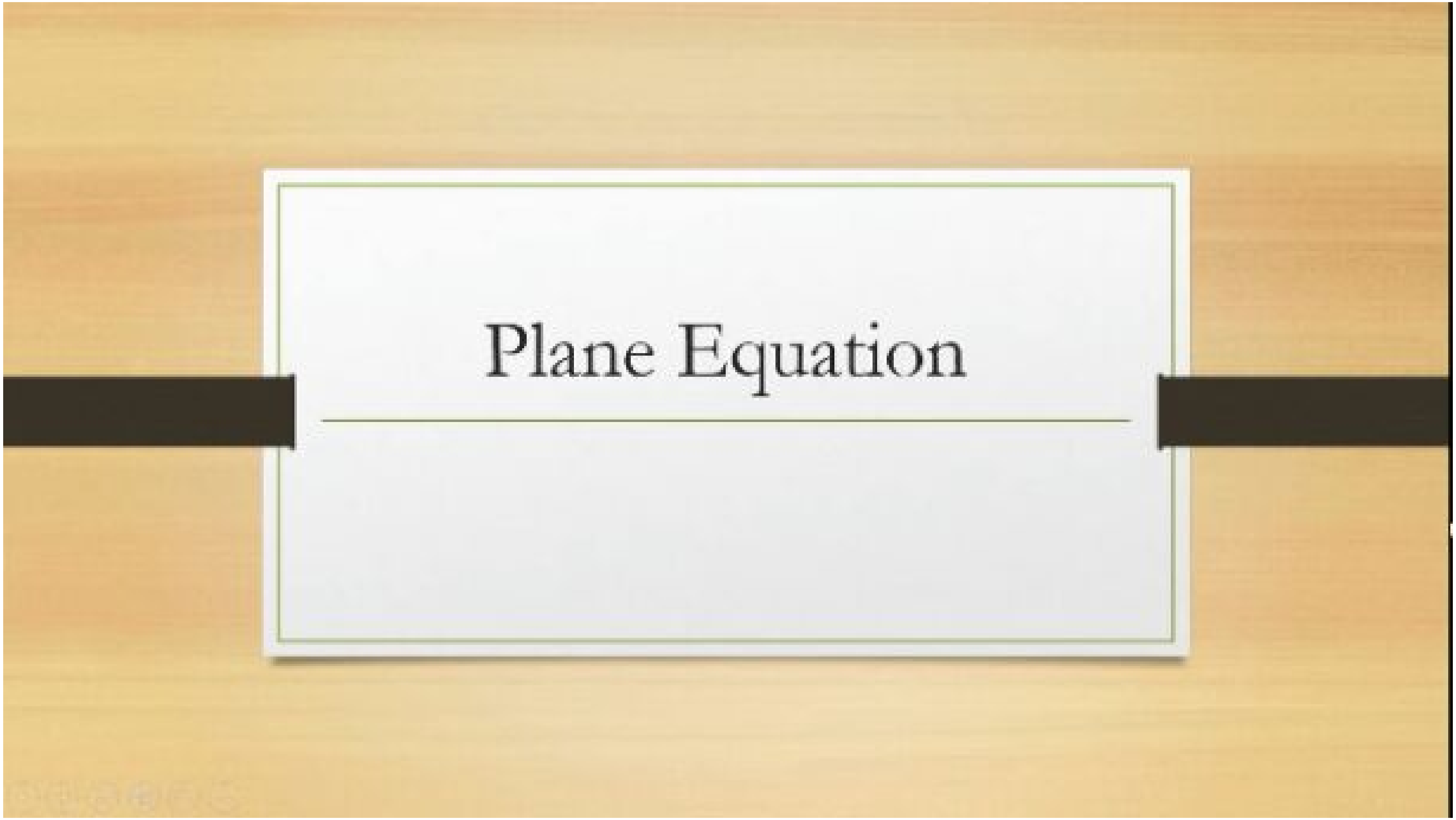
- The angle between 2 lines is the angle between the 2 direction vectors.
- Find the acute angle between $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$
- $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 2 + 3(2) + 4(-4) = -8$
- $\left|\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}\right| = \sqrt{29}, \left|\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}\right| = \sqrt{21}$
- $\cos\theta = \frac{8}{\sqrt{29}\sqrt{21}} \Rightarrow \theta = 71.1°$


Distance between point and line

Distance between point and line

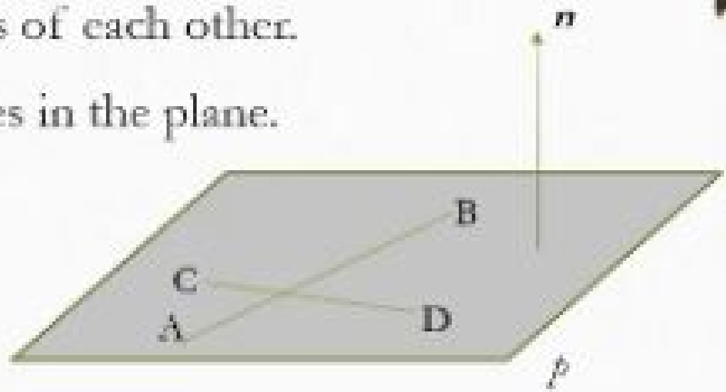
- Find the distance between point M(0, 2, 3) and line $r = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$
- From M, draw a perpendicular to the line and it meets the line at P
- Since P is on the line, vector p is in the form of $\begin{pmatrix} 3+2s \\ 1+s \\ -1+2s \end{pmatrix}$
- $\overrightarrow{MP} = p - m = \begin{pmatrix} 3+2s \\ 1+s \\ -1+2s \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3+2s \\ -1+s \\ -4+2s \end{pmatrix}$
- Since MP ⊥ l, $\begin{pmatrix} 3+2s \\ -1+s \\ -4+2s \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0$, $6+4s-1+s-8+4s=0$. So $s = \frac{1}{3}$.
- So the distance is $|\overrightarrow{MP}| = \left|\begin{pmatrix} 3+2\times\frac{1}{3} \\ -1+\frac{1}{3} \\ -4+2\times\frac{1}{3} \end{pmatrix}\right| = \sqrt{(\frac{11}{3})^2+(\frac{2}{3})^2+(\frac{10}{3})^2} = 5$




Plane Equation

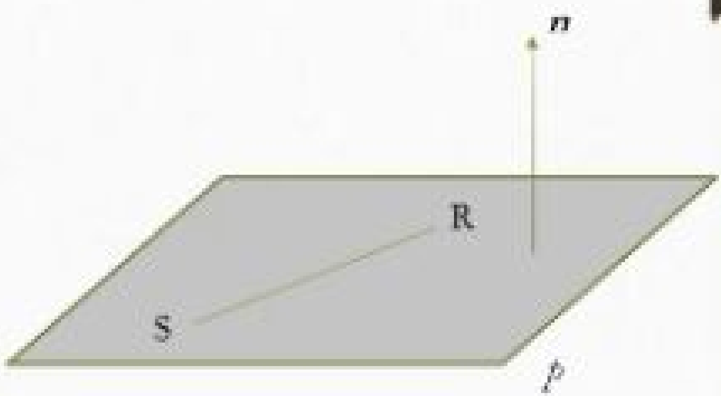
Normal to a Plane

- Vector perpendicular to a plane is called normal to the plane.
- There are many normals to a plane.
- They are all parallel to each other and multiples of each other.
- Normal to a plane is perpendicular to ALL lines in the plane.
- $\boldsymbol{n} \perp AB$. $\boldsymbol{n} \perp CD$.



Plane Equation

- Given a point S in the plane and normal to the plane $\boldsymbol{n}$
- For any point R in plane, $\boldsymbol{n} \perp SR$.
- $\boldsymbol{n}.\overrightarrow{SR} = 0 \quad \boldsymbol{n}.(\boldsymbol{r} - \boldsymbol{s}) = 0 \quad \boldsymbol{n}.\boldsymbol{r} = \boldsymbol{n}.\boldsymbol{s}$
- $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \boldsymbol{n}.\boldsymbol{s} = d$
- $ax + by + cz = d$

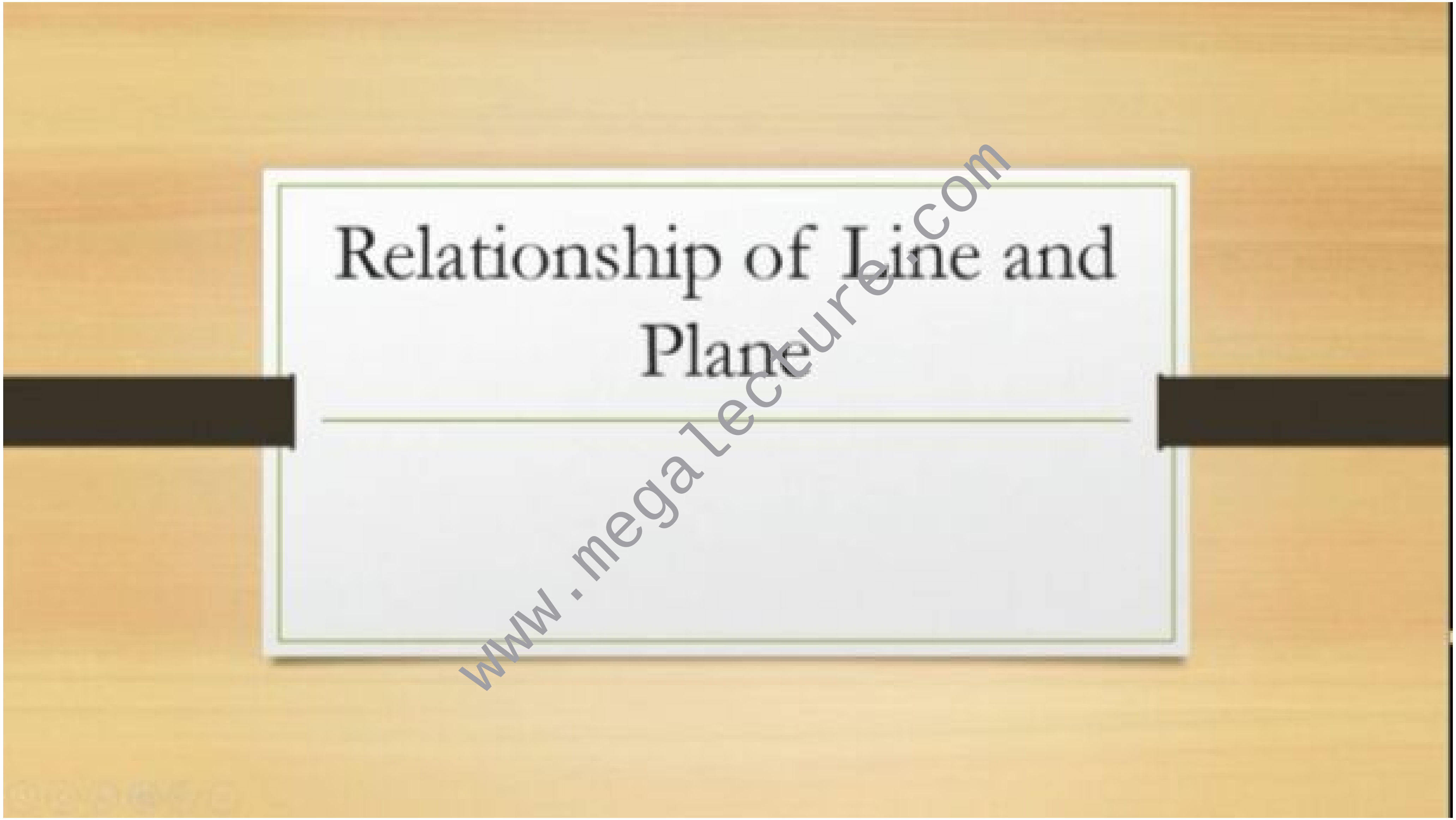


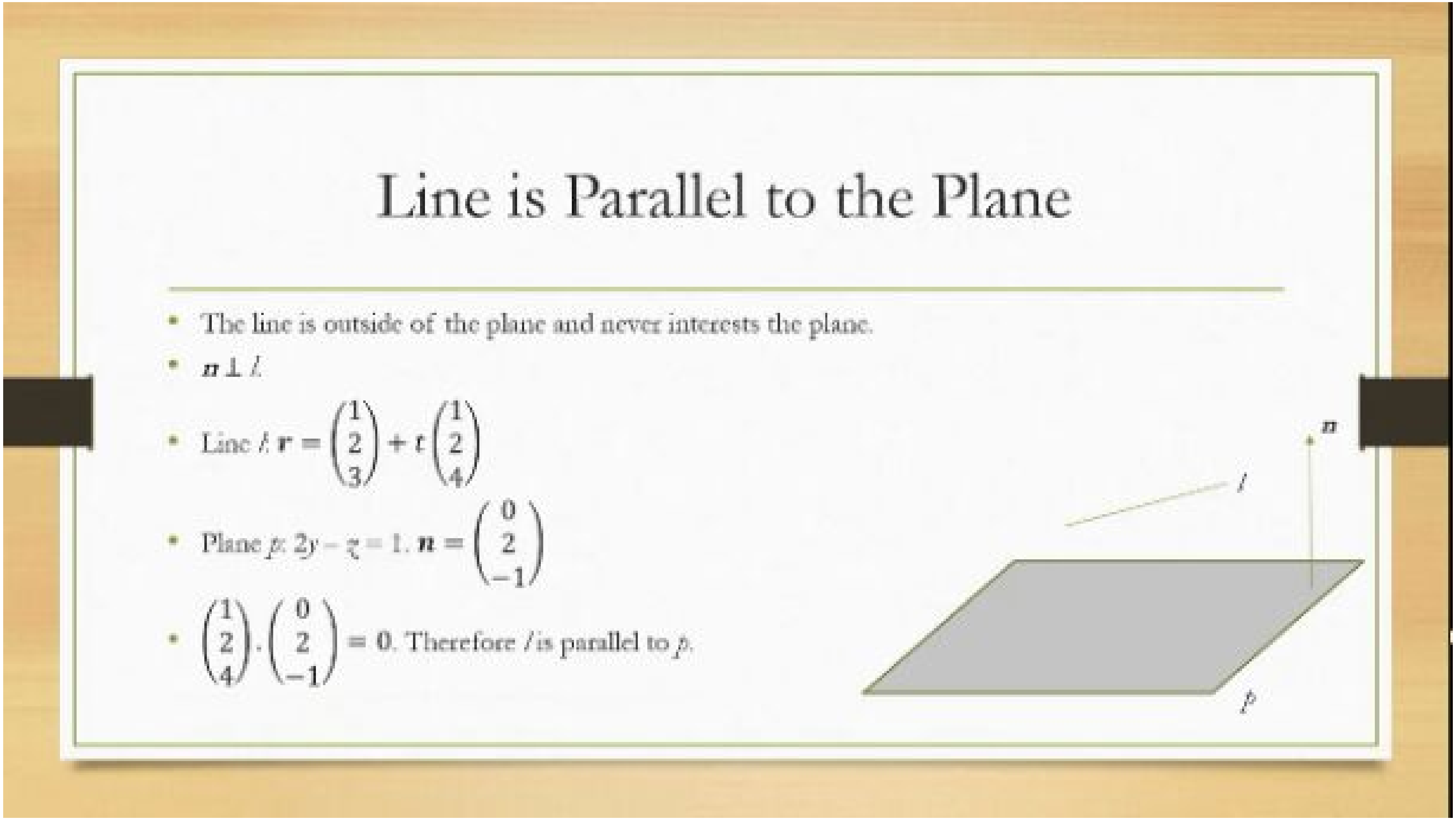
Example

- Find the equation of a plane that passes point A(1, 2, 3) and whose normal is $\begin{pmatrix}4\\5\\6\end{pmatrix}$.
- The plane equation is $4x + 5y + 6z = d$.
- $d = \begin{pmatrix}1\\2\\3\end{pmatrix} . \begin{pmatrix}4\\5\\6\end{pmatrix} = 32$
- $4x + 5y + 6z = 32$

Parallel Planes

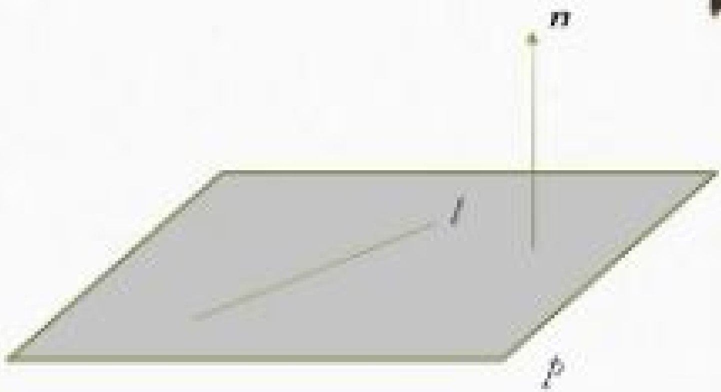
- Planes are parallel if their normals are parallel.
- $4x + 5y + 6z = 3$
- $4x + 5y + 6z = 5$
- $8x + 10y + 12z = 3$


Relationship of Line and
Plane


Line is Parallel to the Plane
The line is outside of the plane and never interests the plane.
$\boldsymbol{n} \perp l$
Line l: $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\2\\4\end{pmatrix}$
Plane p: $2y - z = 1$, $\boldsymbol{n} = \begin{pmatrix}0\\2\\-1\end{pmatrix}$
$\begin{pmatrix}1\\2\\4\end{pmatrix} \cdot \begin{pmatrix}0\\2\\-1\end{pmatrix} = 0$. Therefore l is parallel to p.
n
l
p

Line is in the Plane

- $n \perp l$
- Substitute the line equation into the plane equation and it's always true.
- Line l: $\boldsymbol{r} = \begin{pmatrix}1\\3\\5\end{pmatrix} + t\begin{pmatrix}1\\2\\4\end{pmatrix}$ Plane p: $2y - z = 1$.
- $\begin{pmatrix}1\\2\\4\end{pmatrix} \cdot \begin{pmatrix}0\\2\\-1\end{pmatrix} = 0$
- $\boldsymbol{r} = \begin{pmatrix}1+t\\3+2t\\5+4t\end{pmatrix}$ LHS $= 2(3+2t) - (5+4t) = 1 =$ RHS. Always true.
- So l is in p.



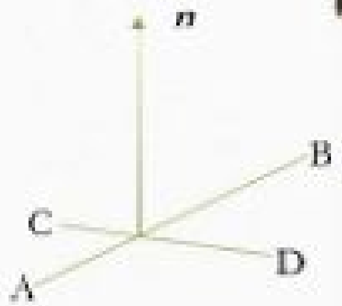
Line Intersects Plane

- Find the intersection of line $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}1\\2\\3\end{pmatrix}$ and plane $2x + 3y + 4z = 5$.
- $\boldsymbol{r} = \begin{pmatrix}1+t\\2+2t\\3+3t\end{pmatrix}$
- Substitute in the plane equation: $2(1 + t) + 3(2 + 2t) + 4(3 + 3t) = 5$.
- $t = -\frac{3}{4}$. Intersection is $\left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$.


Cross Product

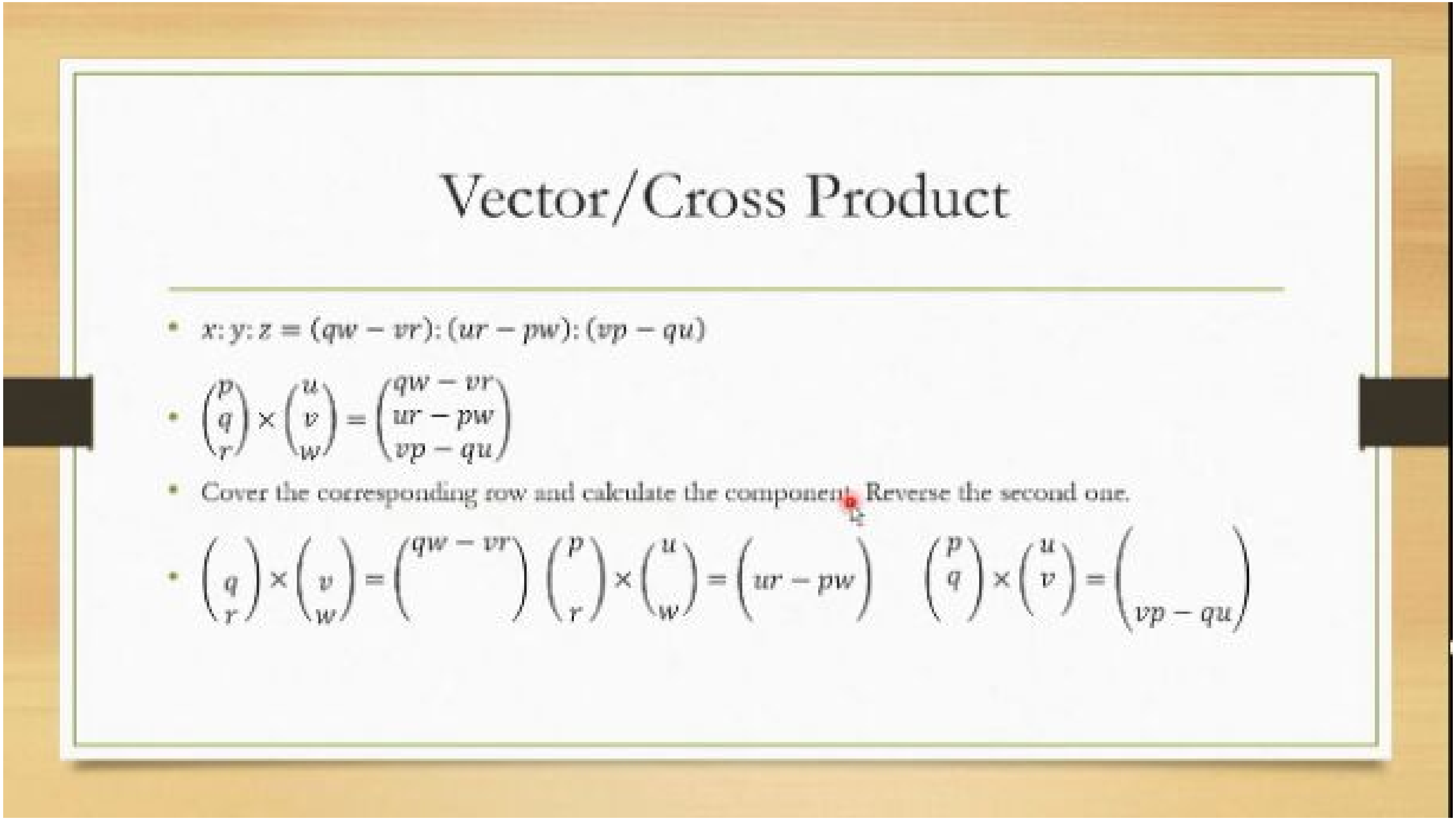
Vector/Cross Product

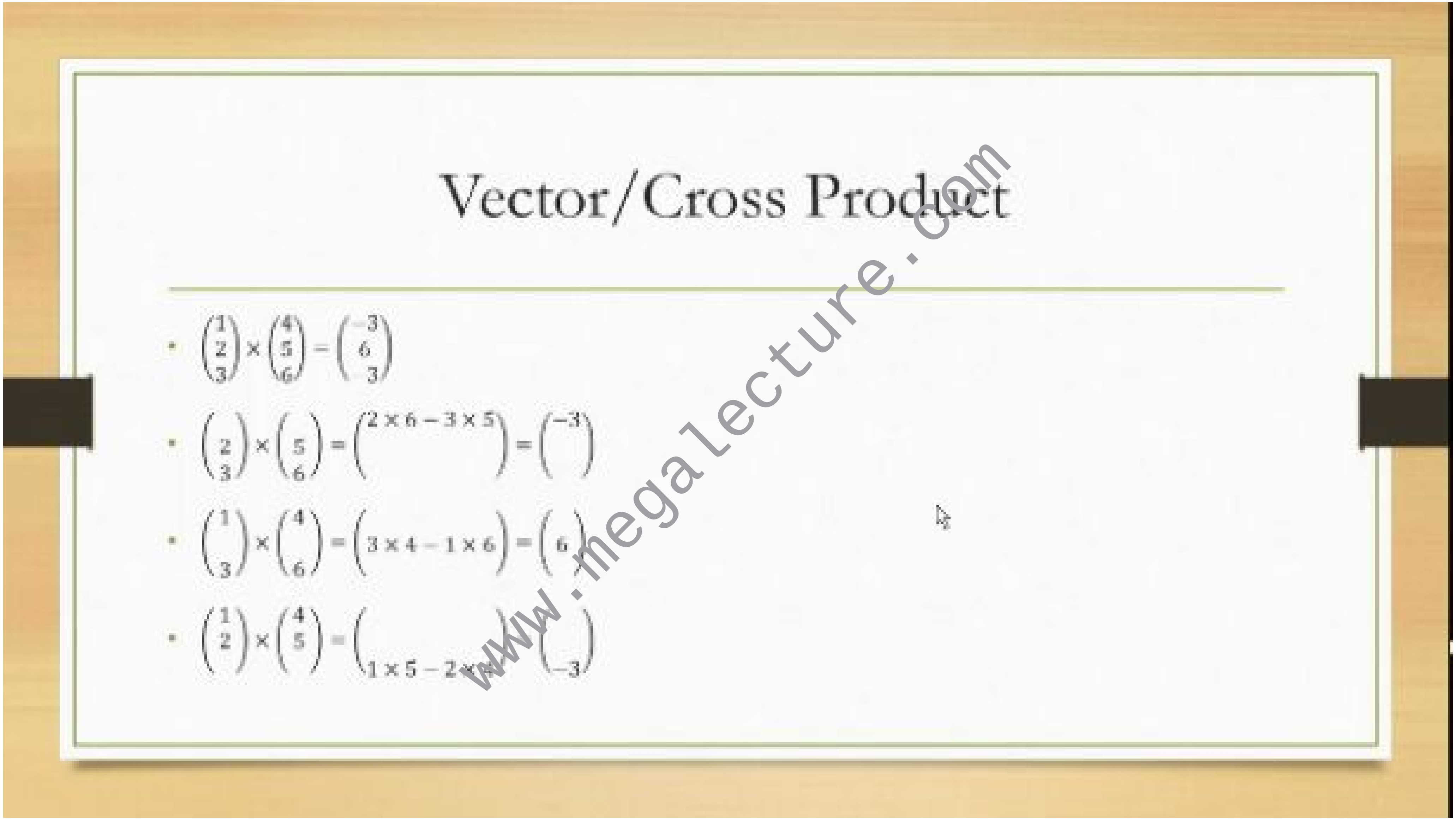
- The vector/cross product of 2 vectors is a vector that is perpendicular to both. It is a common perpendicular to both vectors.
- $\overrightarrow{AB} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \overrightarrow{CD} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \boldsymbol{n} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- $\overrightarrow{AB}.\boldsymbol{n} = 0, \overrightarrow{CD}.\boldsymbol{n} = 0$
- $\begin{cases} px + qy + rz = 0 \\ ux + vy + wz = 0 \end{cases}$



Vector/Cross Product

- $\begin{cases} px + qy + rz = 0 & \times w \\ ux + vy + wz = 0 & \times r \end{cases}$ $\quad \begin{cases} pwx + qwy + rwz = 0 \\ urx + vry + wrz = 0 \end{cases}$
- $(ur - pw)x = (qw - vr)y \qquad x : y = (qw - vr) : (ur - pw)$
- $\begin{cases} px + qy + rz = 0 & \times v \\ ux + vy + wz = 0 & \times q \end{cases}$ $\quad \begin{cases} vpx + vqy + vrz = 0 \\ qux + qvy + qwz = 0 \end{cases}$
- $(vp - qu)x = (qw - vr)z \qquad x : z = (qw - vr) : (vp - qu)$
- $x : y : z = (qw - vr) : (ur - pw) : (vp - qu)$

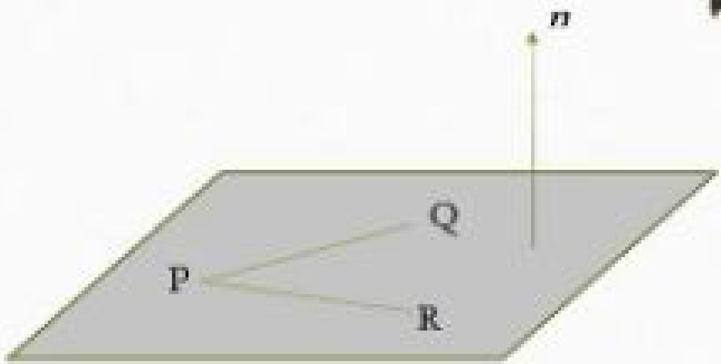

Vector/Cross Product
$x : y : z = (qw - vr) : (ur - pw) : (vp - qu)$
$\begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} qw - vr \\ ur - pw \\ vp - qu \end{pmatrix}$
Cover the corresponding row and calculate the component. Reverse the second one.
$\begin{pmatrix} \\ q \\ r \end{pmatrix} \times \begin{pmatrix} \\ v \\ w \end{pmatrix} = \begin{pmatrix} qw - vr \\ \\ \end{pmatrix}$ $\begin{pmatrix} p \\ \\ r \end{pmatrix} \times \begin{pmatrix} u \\ \\ w \end{pmatrix} = \begin{pmatrix} \\ ur - pw \\ \end{pmatrix}$ $\begin{pmatrix} p \\ q \\ \end{pmatrix} \times \begin{pmatrix} u \\ v \\ \end{pmatrix} = \begin{pmatrix} \\ \\ vp - qu \end{pmatrix}$


Vector/Cross Product
$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix}-3\\6\\-3\end{pmatrix}$
$\begin{pmatrix}\\2\\3\end{pmatrix} \times \begin{pmatrix}\\5\\6\end{pmatrix} = \begin{pmatrix}2\times6-3\times5\\\\\end{pmatrix} = \begin{pmatrix}-3\\\\\end{pmatrix}$
$\begin{pmatrix}1\\\\3\end{pmatrix} \times \begin{pmatrix}4\\\\6\end{pmatrix} = \begin{pmatrix}\\3\times4-1\times6\\\end{pmatrix} = \begin{pmatrix}\\6\\\end{pmatrix}$
$\begin{pmatrix}1\\2\\\end{pmatrix} \times \begin{pmatrix}4\\5\\\end{pmatrix} = \begin{pmatrix}\\\\1\times5-2\times4\end{pmatrix} = \begin{pmatrix}\\\\-3\end{pmatrix}$


Topics Related to Cross Product

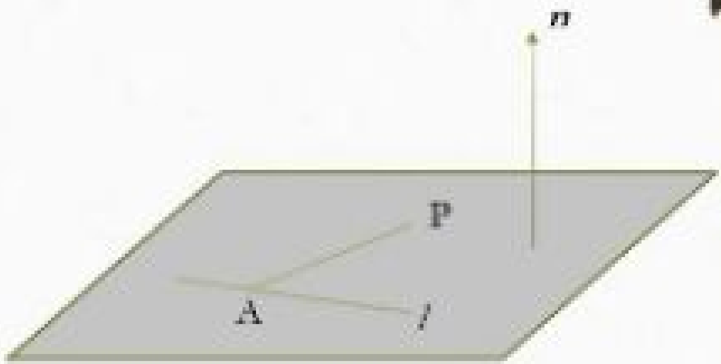
Find plane given 3 points

- Determine the equation of the plane that contains the points P(1, −2, 0), Q(3, 1, 4) and R(0, −1, 2).
- $n \perp PQ$, $n \perp PR$.
- $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \overrightarrow{PR} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$
- $\boldsymbol{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \\ 5 \end{pmatrix}$
- The plane equation is $2x - 8y + 5z = d$.
- Substitute P into the equation: $2(1) - 8(-2) + 5(0) = d = 18$.
- The plane equation is $2x - 8y + 5z = 18$.



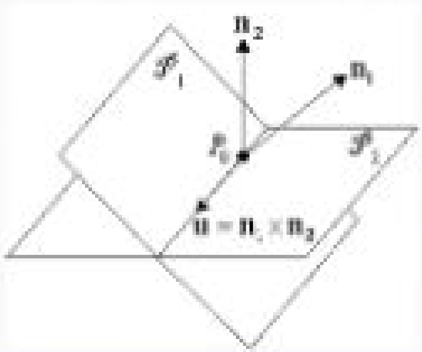
Find plane given a point and a line

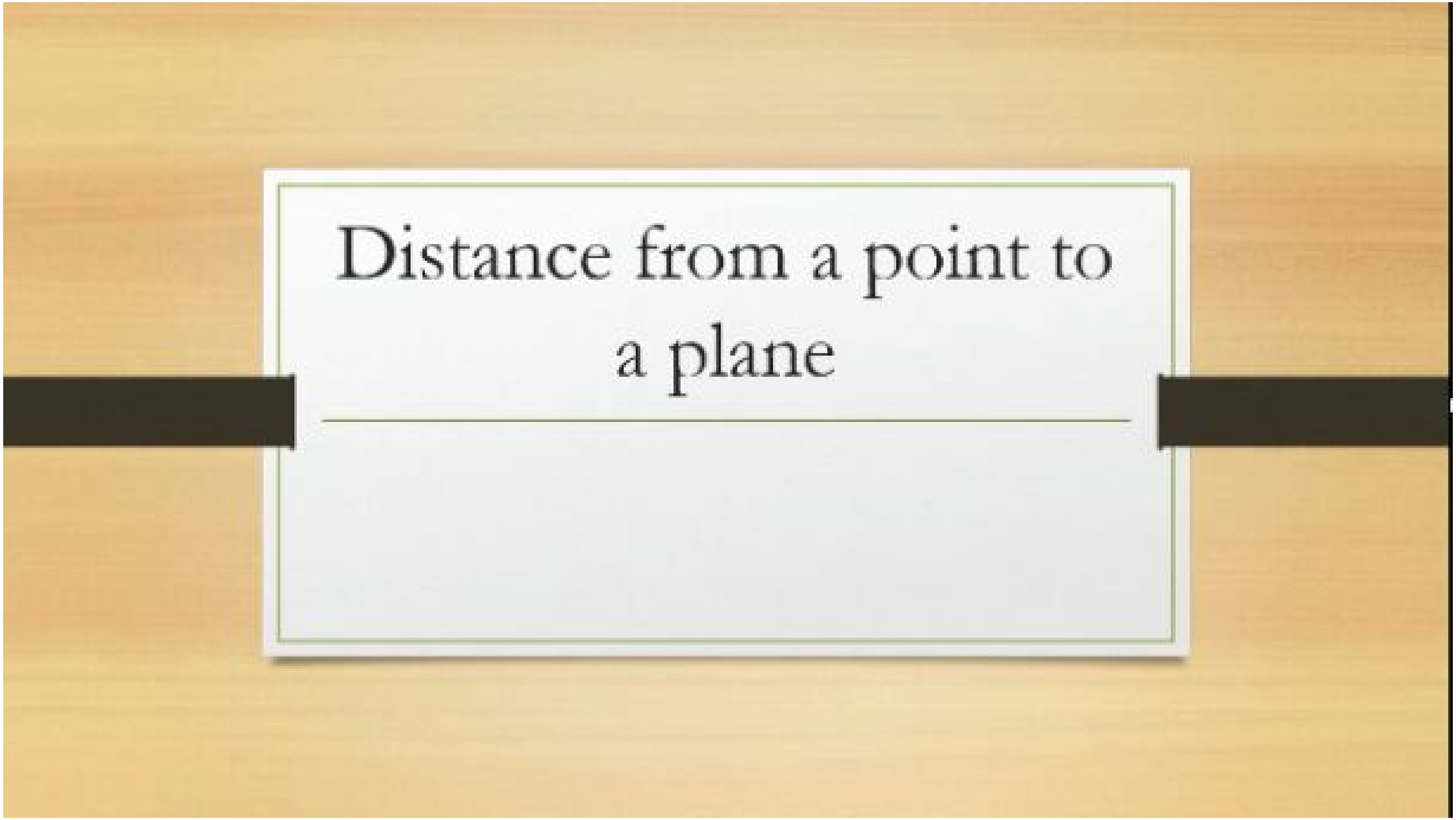
- Determine the equation of the plane that passes through the point P(1, 6, −4) and contains the line $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\-1\end{pmatrix}$
- $\mathbf{n} \perp AP,\ \mathbf{n} \perp l$
- $\overrightarrow{AP} = \begin{pmatrix}1\\6\\-4\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}0\\4\\-7\end{pmatrix}$
- $\mathbf{n} = \overrightarrow{AP} \times \textit{direction vector} = \begin{pmatrix}0\\4\\-7\end{pmatrix} \times \begin{pmatrix}2\\-3\\-1\end{pmatrix} = \begin{pmatrix}-25\\-14\\-8\end{pmatrix} = -\begin{pmatrix}25\\14\\8\end{pmatrix}$
- The plane equation is $25x + 14y + 8z = d$
- Substitute P into the equation: $25(1) + 14(6) + 8(-4) = d = 18.$
- The plane equation is $25x + 14y + 8z = 77.$



Find intersection line between 2 planes

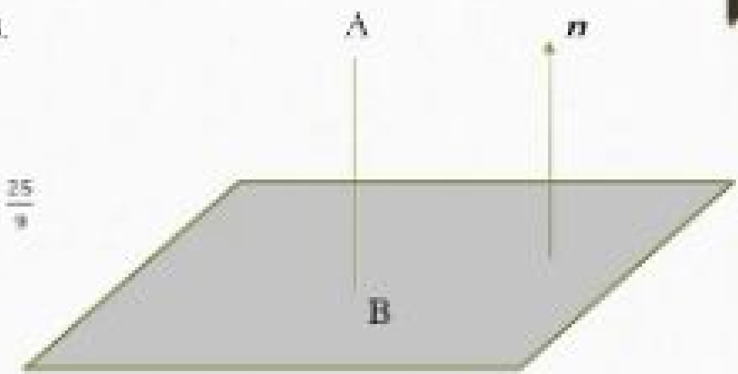
- Find the intersection line equation of $P_1: x + 2y + 3z = 4$ and $P_2: 5x + 6y + 7z = 8$.
- The intersection line is perpendicular to the two normals.
- So the direction vector is $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}5\\6\\7\end{pmatrix} = \begin{pmatrix}-4\\8\\-4\end{pmatrix} = -4\begin{pmatrix}1\\-2\\1\end{pmatrix}$
- $\begin{cases} x + 2y + 3z = 4 \\ 5x + 6y + 7z = 8 \end{cases}$ Let $x = 0$, $\begin{cases} 2y + 3z = 4 \\ 6y + 7z = 8 \end{cases} \Rightarrow \begin{cases} y = -1 \\ z = 2 \end{cases}$
- The line equation is $\boldsymbol{r} = \begin{pmatrix}0\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\1\end{pmatrix}$




Distance from a point to a plane

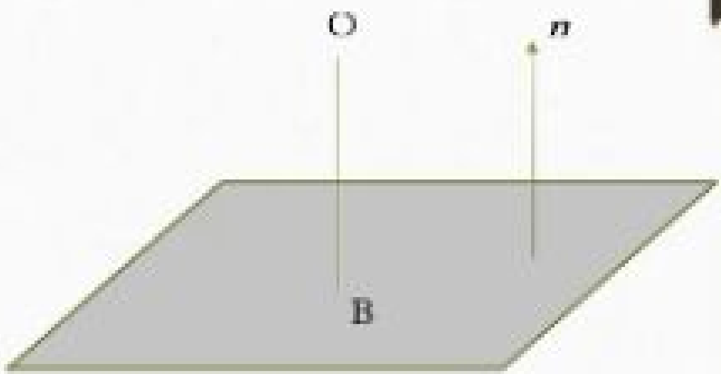
Distance from a point to a plane

- Find the distance from the point $A(2, 8, 5)$ to the plane $x - 2y - 2z = 1$
- $\boldsymbol{n} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$
- From A, make a perpendicular to the plane which intersects the plane at B.
- The equation of AB is: $\boldsymbol{r} = \begin{pmatrix} 2 \\ 8 \\ 5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2+t \\ 8-2t \\ 5-2t \end{pmatrix}$
- Substitute it into the plane equation: $2 + t - 2(8 - 2t) - 2(5 - 2t) = 1 \Rightarrow t = \frac{25}{9}$
- Distance is $|\overline{AB}| = \left| t\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \right| = \frac{25}{9}\sqrt{1^2 + 2^2 + 2^2} = \frac{25}{3}$



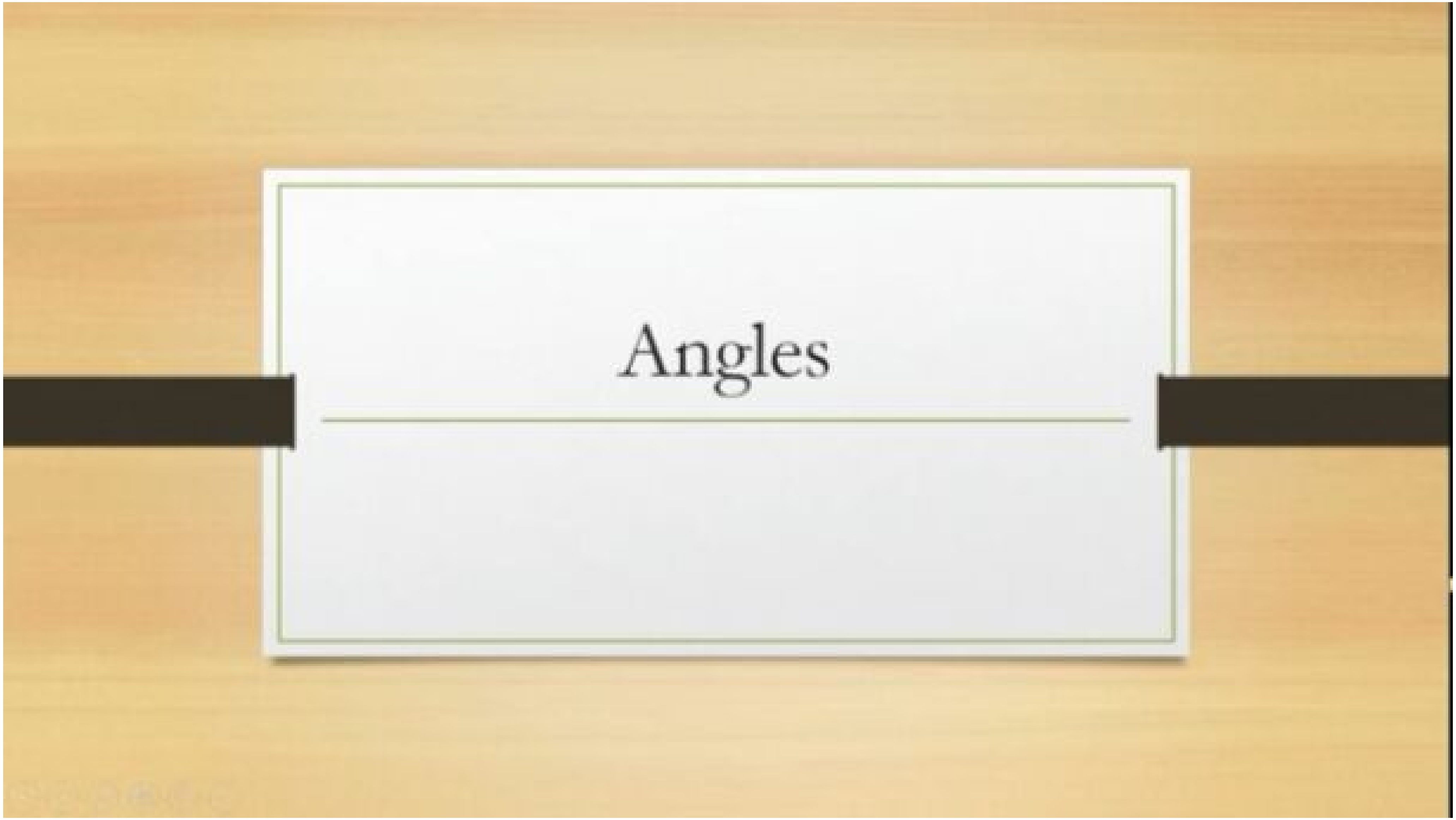
Distance from origin O to a plane

- Find the distance from the origin to plane $ax + by + cz = d$.
- $n = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$
- From O, make a perpendicular to the plane which intersects the plane at B.
- The equation of OB is: $r = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow \begin{pmatrix} at \\ bt \\ ct \end{pmatrix}$
- Substitute it into the plane equation: $a^2t + b^2t + c^2t = d \Rightarrow t = \frac{d}{a^2+b^2+c^2}$
- Distance is $|\overrightarrow{OB}| = \left|t\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right| = \frac{|d|}{a^2+b^2+c^2}\sqrt{a^2+b^2+c^2} = \frac{|d|}{\sqrt{a^2+b^2+c^2}}$



Distance between 2 parallel planes

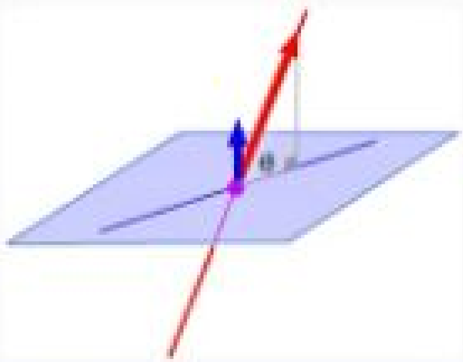
- The distance between 2 parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is $\frac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}}$


Angles

MEGA LECTURE

Angle between a Line and a Plane

- The angle between a line and a plane is $90°$ − the acute angle between the direction vector of the line and the normal of the plane.
- Determine the angle between the line $\boldsymbol{r} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and the plane $-x + y = 1$.
- direction vector is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\boldsymbol{n} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.
- $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1$ We take the positive value 1.
- The angle between them is $\cos^{-1}\frac{1}{\sqrt{2^2+1^2+2^2} \times \sqrt{(-1)^2+1^2+0^2}} = \cos^{-1}\frac{1}{3\sqrt{2}} \approx 76.37°$
- So the angle between the line and the plane is $90 - 76.37 = 13.6°$



Angle between 2 Planes

- The acute angle between 2 planes is the acute angle between 2 normals.
- Find the acute angle between $x + y + z = 1$ and $x + y - 3z = 2$.
- $\begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-3\end{pmatrix} = -1$ We take the positive value 1.
- The acute angle between the 2 planes is
- $\cos^{-1}\frac{1}{\sqrt{1^2+1^2+1^2}\times\sqrt{1^2+1^2+3^2}} = \cos^{-1}\frac{1}{\sqrt{33}} \approx 80.0°$