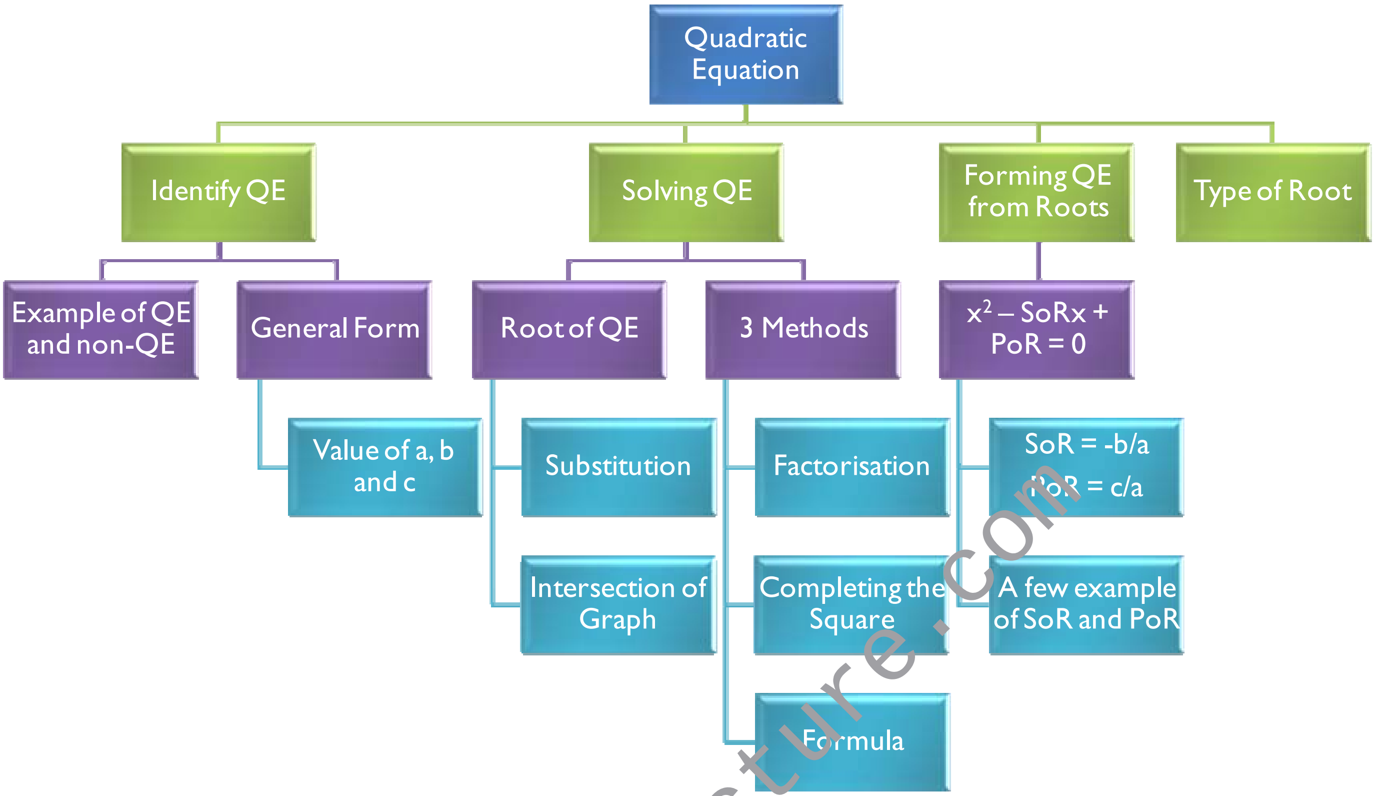
Quadratic Equation

Mind Map



Quadratic Equation

In mathematics, a quadratic equation is a polynomial equation of the second degree. The general form is

$$ax^2+bx+c=0$$

where a ≠ 0. (For a = 0, the equation becomes a linear equation.)

The letters a, b, and c are called coefficients: the quadratic coefficient a is the coefficient of x2, the linear coefficient b is the coefficient of x, and c is the constant coefficient, also called the free term or constant term.

Example of Quadratic Equation

$2x^2-5=0$

$1-6x^2=3$

$6x+3x^2=0$

$x^2=0$

Difference Between Quadratic Eqation and Quadratic Function

Quadratic Functions	*Quadratic Equations*
$y=x^2-3x+2$ $f(x)=x^2-3x+2$	$x^2-3x+2=0$

Quadratic Equation

Solving Quadratic Equation

3 Methods:

- Factorisation
- Completing The Square
- Quadratic Formula

Factorisation

Topic 2,3 Quadratic Equations & Functions:::::::::: v 2,3.1 Find the roots ::::::: v 2,3.11 Factorisation

e.g.1:

$$x^2-3x+2=0$$
$$(x-1)(x-2)=0$$
$$x=1 @ 2 \#$$

×	×	+
x	-1	$-x$
x	-2	$-2x$
x^2	$+2$	$-3x$

e.g.2:

$$x^2+2x-360=0$$
$$(x+20)(x-18)=0$$
$$x=18 @ -20 \#$$

×	×	+
x	$+20$	$-18x$
x	-18	$+20x$
x^2	-360	$+2x$

Example

Solve $x^2 + 5x + 6 = 0$.

Answer

$x^2 + 5x + 6 = (x + 2)(x + 3)$

Set this equal to zero:

$(x + 2)(x + 3) = 0$

Solve each factor:

$x + 2 = 0$ *or* $x + 3 = 0$

$x = -2$ or $x = -3$

The solution of $x2 + 5x + 6 = 0$ is $x = -3, -2$

Completing The Square

e.g.1:

Solve the equation $x^2-2x-5=0$.

Solution:

$$x^2-2x-5=0$$
$$(x^2-2x)-5=0$$
$$\left[(x-1)^2-(-1)^2\right]-5=0$$
$$(x-1)^2-1-5=0$$
$$(x-1)^2-6=0$$
$$(x-1)^2=6$$
$$x-1=\pm\sqrt{6}$$
$$x=1\pm\sqrt{6}$$
$$x=3.499 @ -1.499 \#$$

e.g.2:

Solve the equation $x^2-6x-10=0$.

Solution:

$$x^2-6x-10=0$$
$$(x^2-6x)-10=0$$
$$\left[(x-3)^2-(-3)^2\right]-10=0$$
$$(x-3)^2-9-10=0$$
$$(x-3)^2-19=0$$
$$(x-3)^2=19$$
$$x-3=\pm\sqrt{19}$$
$$x=3\pm\sqrt{19}$$
$$x=7.359 @ -1.359 \#$$

Quadratic Equation

Quadratic Formula

Topic 2,3 Quadratic Equations & Functions:::::::::: v 2,3.1 Find the roots ::::::: v 2,3.13 The Method Of Formulae

The formula to find roots, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Prove that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ if $ax^2 + bx + c = 0$.

$$ax^2 + bx + c = 0$$
$$(\div a)$$
$$\left(x^2 + \frac{b}{a}x\right) + \left(\frac{c}{a}\right) = 0$$
$$\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + \left(\frac{c}{a}\right) = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \left(\frac{c}{a}\right) = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2}\right) + \left(\frac{c}{a}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2}{4a^2}\right) - \left(\frac{c}{a}\right)$$
$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b^2 - 4ac}{4a^2}\right)$$
$$\left(x + \frac{b}{2a}\right) = \sqrt{\left(\frac{b^2 - 4ac}{4a^2}\right)}$$
$$\left(x + \frac{b}{2a}\right) = \pm\left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)$$

$$x = -\frac{b}{2a} \pm \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)$$
$$x = \left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right) \#$$

e.g. 1:

Find the roots for the quadratic equation $2x^2 - x - 7 = 0$.

Solution:

$$2x^2 - x - 7 = 0$$
$$a = 2, b = -1, c = -7$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-7)}}{2(2)}$$
$$x = \frac{1 \pm \sqrt{57}}{1}$$
$$x = -1.637 @ 2.137 \#$$

e.g. 2:

Find the roots for the quadratic equation $3x^2 - 3 = 4x$.

Solution:

$$3x^2 - 4x - 3 = 0$$
$$a = 3, b = -4, c = -3$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-3)}}{2(3)}$$
$$x = \frac{4 \pm \sqrt{52}}{6}$$
$$x = 1.869 @ -0.5351 \#$$

Quadratic Equation

Forming Quadratic Equation from Its Roots

Topic 2,3 Quadratic Equations & Functions:::::::::: v 2.2 Form A Quadratic Equation, Solving Problem, & Using the Formula [SOR & POR]:::: v 2.21 Given Roots

*e.g.*1: Form a quadratic equation whose roots are 1 and 2.

Solution:

$x = 1 \quad | \quad x = 2$

$x - 1 = 0 \quad | \quad x - 2 = 0$

$(x-1)(x-2) = 0$

$x^2 - 3x + 2 = 0$ #

Attention: This question is using an inverse concept of factorisation.

e.g.:

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

$x = 1$ @ 2 #

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*e.g.*2: Form a quadratic equation whose roots are $\frac{1}{3}$ and $-\frac{1}{4}$.

Solution:

$x = \frac{1}{3} \quad | \quad x = -\frac{1}{4}$

$3x = 1 \quad | \quad 4x = -1$

$3x - 1 = 0 \quad | \quad 4x + 1 = 0$

$(3x-1)(4x+1) = 0$

$12x^2 - x - 1 = 0$ #

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*e.g.*3: Form a quadratic equation whose roots are 0 and -5.

Solution:

$x = 0 \quad | \quad x = -5$

$\quad | \quad x + 5 = 0$

$x(x+5) = 0$

$x^2 + 5x = 0$ #

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*e.g.*4: Form a quadratic equation whose roots are $\frac{2}{3}$ and -1.

Solution:

$x = \frac{2}{3} \qquad x = -1$

$3x = 2 \qquad x + 1 = 0$

$3x - 2 = 0$

$(3x-2)(x+1) = 0$

$3x^2 + x - 2 = 0$ #

Quadratic Equation

e.g.2:

If α and β are the roots for the quadratic equation $x^2+3x+5=0$, Form a quadratic equation that has the following roots:

a) $\frac{\alpha}{2},\frac{\beta}{2}$ b) $(2\alpha-1),(2\beta-1)$

Before we solve the (a) and (b), we must find the SOR and POR.

$x^2+3x+5=0$

$a=1, b=3, c=5$

roots : α,β

SOR : $\alpha+\beta=-\frac{b}{a}$, $\alpha+\beta=-\frac{(3)}{1}$, $\alpha+\beta=-3$

POR : $\alpha\beta=\frac{c}{a}$, $\alpha\beta=\frac{(5)}{1}$, $\alpha\beta=5$

(a) Solution:

Let the new equation as $x^2+Bx+C=0$,
$a=1, b=B, c=C$

New Roots : $\frac{\alpha}{2},\frac{\beta}{2}$

SOR: $\frac{\alpha}{2}+\frac{\beta}{2}=-\frac{B}{1}$

$\frac{(\alpha+\beta)}{2}=-B$

$\frac{(-3)}{2}=-B$

$B=\frac{3}{2}$

POR: $\frac{\alpha}{2}\times\frac{\beta}{2}=\frac{C}{1}$

$\frac{\alpha\beta}{4}=C$

$\frac{5}{4}=C$

$C=\frac{5}{4}$

$x^2+\frac{3}{2}x+\frac{5}{4}=0$

$(\times 4)$

$4x^2+6x+5=0$

$\therefore$ The quadratic equation that has the roots $\frac{\alpha}{2},\frac{\beta}{2}$ is $4x^2+6x+5=0$#

** The quadratic equation can not has fraction

(b)

Let the new equation as $x^2+Bx+C=0$,
$a=1, b=B, c=C$

New Roots : $(2\alpha-1),(2\beta-1)$

SOR: $(2\alpha-1)+(2\beta-1)=-\frac{B}{1}$

$2(\alpha+\beta)-2=-B$

$2(-3)-2=-B$

$-B=-8$

$B=8$

POR: $(2\alpha-1)(2\beta-1)=\frac{C}{1}$

$4\alpha\beta-2(\alpha+\beta)+1=C$

$4(5)-2(-3)+1=C$

$C=27$

$\therefore$ The quadratic equation that has the roots $(2\alpha-1),(2\beta-1)$ is $x^2+8x+27=0$#

e.g.3:

If α and β are the roots for the quadratic equation $x^2+2x-5=0$, Form a quadratic equation that has the roots:

***a) α^2,β^2 b) $\frac{1}{\alpha},\frac{1}{\beta}$ c) $\frac{\alpha}{\beta},\frac{\beta}{\alpha}$

Before we solve the (a) and (b), we must find the SOR and POR.

$x^2+2x-5=0$

$a=1, b=2, c=-5$

roots : α,β

SOR : $\alpha+\beta=-\frac{b}{a}$, $\alpha+\beta=-\frac{(2)}{1}$, $\alpha+\beta=-2$

POR : $\alpha\beta=\frac{c}{a}$, $\alpha\beta=\frac{(-5)}{1}$, $\alpha\beta=-5$

Solution:

****(a)

Let the new equation as $x^2+Bx+C=0$,
$a=1, b=B, c=C$

New Roots : α^2,β^2

$(\alpha+\beta)^2=\alpha^2+2\alpha\beta+\beta^2$

$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

Use this formula if has $\alpha^2+\beta^2$

SOR: $\alpha^2+\beta^2=-\frac{B}{1}$

$(\alpha+\beta)^2-2\alpha\beta=-B$

$(-2)^2-2(-5)=-B$

$-B=14$

$B=14$

POR: $\alpha^2\times\beta^2=\frac{C}{1}$

$(\alpha\beta)^2=C$

$(-5)^2=C$

$C=25$

$\therefore$ The quadratic equation that has the roots α^2,β^2 is $x^2-14x+25=0$#

Quadratic Equation

(b) New Roots: $\frac{1}{\alpha}, \frac{1}{\beta}$

$\alpha+\beta=-2, \alpha\beta=-5$

Let the quadratic equation as $x^2+Bx+C=0$.

SOR: $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{B}{1}$

$\frac{\alpha+\beta}{\alpha\beta}=-B$

$\frac{(-2)}{(-5)}=-B$

$-B=\frac{2}{5}$

$B=-\frac{2}{5}$

POR: $\frac{1}{\alpha}\times\frac{1}{\beta}=\frac{C}{1}$

$\frac{1}{\alpha\beta}=C$

$\frac{1}{(-5)}=C$

$C=-\frac{1}{5}$

$x^2-\frac{2}{5}x-\frac{1}{5}=0$

$5x^2-2x-1=0$

$\therefore$ The quadratic equation that has the roots $\frac{1}{\alpha}, \frac{1}{\beta}$ is $5x^2-2x-1=0$ #

(c) New Roots: $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

$\alpha+\beta=-2, \alpha\beta=-5$

Let the quadratic equation as $x^2+Bx+C=0$.

SOR: $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-\frac{B}{1}$

$\frac{\alpha^2+\beta^2}{\alpha\beta}=-B$

$\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=-B$

$-B=\frac{(-2)^2-2(-5)}{(-5)}$

$B=\frac{14}{5}$

POR: $\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=\frac{C}{1}$

$\frac{\alpha\beta}{\alpha\beta}=C$

$\frac{(-5)}{(-5)}=C$

$C=1$

$x^2+\frac{14}{5}x+1=0$

$5x^2+14x+5=0$

$\therefore$ The quadratic equation that has the roots $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ is $5x^2+14x+5=0$ #

Trick Questions

1. If α and β are the roots for the quadratic equation $x^2-2x-3=0$, Form a quadratic equation that has the roots $\frac{\alpha}{\beta^2}$ and $\frac{\beta}{\alpha^2}$ $(\alpha>\beta)$.

**$(\alpha>\beta)\rightarrow$ can calculate the α and β.

Usually we can find the value of x.

Solution:

$x^2-2x-3=0$

$(x-3)(x+1)=0$

$x=-1 @ 3$

Given that $\alpha>\beta, \therefore \alpha=3, \beta=-1$

$x=\frac{\alpha}{\beta^2}$

$x=\frac{3}{(-1)^2}$

$x=3$

$x-3=0$

$x=\frac{\beta}{\alpha^2}$

$x=\frac{(-1)}{(3)^2}$

$x=-\frac{1}{9}$

$9x=-1$

$9x+1=0$

$(x-3)(9x+1)=0$

$\therefore x^2-26x-3=0$ #

Quadratic Equation

2. If α and β are the roots for the quadratic equation $x^2 - x - 6 = 0$, Form a quadratic equation that has the roots α and $\frac{1}{\beta}(\alpha > \beta)$.

Solution:

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = -2 @ 3$$

Given that $\alpha > \beta, \therefore \alpha = 3, \beta = -2$

$x = \alpha$, $x = 3$, $x - 3 = 0$

$x = \frac{1}{\beta}$, $x = \frac{1}{(-2)}$, $2x = -1$, $2x + 1 = 0$

$$(x-3)(2x+1) = 0$$
$$\therefore 2x^2 - 5x - 3 = 0 \#$$

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3. The roots for a quadratic equation $2x^2 - (k-2)x - 3k = 0$ are p and q. If $q = 1 - p$, Calculate the possible value(s) of k, p and q.

Solution:

$2x^2 - (k-2)x - 3k = 0$......... [A]
$a = 2, b = -(k-2), c = -3k$
$roots: p, q$

$\underline{SOR}$:
$p + q = -\frac{-(k-2)}{2}$
$p + q = \frac{k-2}{2}$.................. [1]

$\underline{POR}$:
$pq = \frac{-3k}{2}$
$pq = -\frac{3k}{2}$.................. [2]

But $q = 1 - p$,
$p + q = 1$........... [3]

From [1] and [3],
$\frac{k-2}{2} = 1$
$k - 2 = 2$
$k = 4 \#$

Substitute k and [A],
$2x^2 - (4-2)x - 3(4) = 0$
$2x^2 - 2x - 12 = 0$
$x^2 - x - 6 = 0$
$(x-3)(x+2) = 0$
$x = -2 @ 3$

If $p = -2$, then $q = 3$ @ If $p = 3$, then $q = -2 \#$

Quadratic Equation

4. If one of the roots for the quadratic equation $x^2 - ax + b = 0$ is twice the other root. Prove that $9b = 2a^2$.

Solution:

roots : $\alpha, 2\alpha$

SOR:

$\alpha + 2\alpha = -\frac{(-a)}{1}$

$3\alpha = a$

$\alpha = \frac{a}{3}$.............{1}

POR:

$\alpha \times 2\alpha = \frac{(b)}{1}$

$2\alpha^2 = b$

$\alpha^2 = \frac{b}{2}$.............{2}

Substitute α into {2},

$2\left(\frac{a}{3}\right)^2 = b$

$2\left(\frac{a^2}{9}\right) = b$

$2a^2 = 9b$ #

Nature of Roots of a Quadratic Equation

The expression $b^2 - 4ac$ in the general formula is called the ***discriminant*** of the equation, as it determines the type of roots that the equation has.

$b^2 - 4ac > 0 \Leftrightarrow$ two real and distinct roots
$b^2 - 4ac = 0 \Leftrightarrow$ two real and equal roots
$b^2 - 4ac < 0 \Leftrightarrow$ no real roots
$b^2 - 4ac \geq 0 \Leftrightarrow$ the roots are real

e.g. 1:

Find the range of values of k for which the equation $2x^2 + 5x + 3 - k = 0$ has two real distinct roots.

$b^2 - 4ac > 0$

$(5)^2 - 4(2)(3-k) > 0$

$25 - 24 + 8k > 0$

$1 + 8k > 0$

$8k > -1$

$k > \frac{-1}{8}$

e.g. 2:

The roots of $3x^2 + kx + 12 = 0$ are equal. Find k.

$b^2 - 4ac = 0$

$(k)^2 - 4(3)(12) = 0$

$k^2 - 144 = 0$

$k^2 = 144$

$k = \pm\sqrt{144}$

$k = \pm 12$

e.g. 3:

Find the range of values of p for which the equation $x^2 - 2px + p^2 + 5p - 6 = 0$ has no real roots.

$b^2 - 4ac < 0$

$(-2p)^2 - 4(1)(p^2 + 5p - 6) < 0$

$4p^2 - 4p^2 - 20p + 24 < 0$

$-20p + 24 < 0$

$-20p < -24$

$20p > 24$

$p > \frac{24}{20}$

$p > \frac{6}{5}$

Quadratic Equation

e.g. 4:

Show that the equation $a^2x^2 + 3ax + 2 = 0$ always has real roots.

$b^2 - 4ac$

$= (3a)^2 - 4(a^2)(2)$

$= 9a^2 - 8a^2$

$= a^2$

$a^2 > 0$ for all values of a. Therefore
$b^2 - 4ac > 0$

Proven that
$a^2x^2 + 3ax + 2 = 0$ always has real roots.