9 POWERS

Objectives

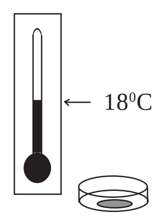
After studying this chapter you should

- understand fractional indices;
- know how to use the binomial theorem for any positive integers;
- be able to answer simple combinational problems.

9.0 Introduction

You are already familiar with expressions like $3^2, 4^{10}$ and 10^{-6}, all of which involve **powers** (or **indices**). But can any meaning be attached to an expression like $2^{0.6}$? If so, does it have any relevance? This chapter starts off by answering these questions. The rest of the chapter is concerned with a famous and important piece of mathematics known as the **binomial theorem**.

The topic is introduced through a case study on bacterial growth. Bacteria perform the roles of friend and foe at the same time. They are micro-organisms that perform a crucial function in nature by causing plant and animal debris to decay in the soil, but at the same time they can cause disease. Under favourable conditions they reproduce freely. Lone bacterium will first split into two bacteria, then both of these bacteria will themselves split into two and so on.



Petri dish with jelly

This growth can be observed by placing a lone bacterium onto a petri dish and positioning the dish in a warm environment. The splitting process under these sorts of conditions will take place twice per hour; hence after one hour there will be 4 bacteria, after two hours 16 bacteria and so on.

Activity 1 Bacterial growth

(a) Copy and complete the table shown. Write a formula for the number of bacteria after t hours.

Time (hours)	0	1	2	3	4	...
Number of bacteria	1	4	16	...	...	...

(b) Interpret this formula when $t = \frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}$.

9.1 Fractional indices

Before continuing, you will find it useful to revise your knowledge of how integer indices work.

Activity 2 Revision

Complete these general statements :

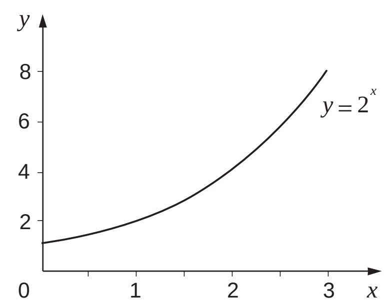
For any non-zero number x, and any integers m and n :

(a) $x^m x^n =$ (b) $\dfrac{x^m}{x^n} =$ (c) $\left(x^m\right)^n =$

(d) $x^0 =$ (e) $x^{-n} =$

Activity 3

(a) You have already seen that $2^0 = 1, 2^1 = 2, 2^2 = 4$ and $2^3 = 8$. Draw an accurate graph of $y = 2^x$ by joining together these points with as smooth a curve as you can.



(b) The value of $2^{1\frac{1}{2}}$ must be double that of $2^{\frac{1}{2}}$. Why? Make a similar statement about $2^{2\frac{1}{2}}$. Are these statements supported by your graph?

(c) What is your interpretation of $2^{\frac{1}{2}}$?

Activity 4 Some logical deductions

(a) Let p stand for the value of $2^{\frac{1}{2}}$. What can you say about p^2? What rule of indices did you use?

(b) What is the meaning of $x^{\frac{1}{2}}$?

(c) Suppose q stands for the value of $2^{\frac{1}{3}}$. What is q^3?

(d) What does $x^{\frac{1}{3}}$ mean?

In general :

$x^{\frac{1}{2}}$ is another way of writing $\sqrt{x}$

$x^{\frac{1}{3}}$ is another way of writing $\sqrt[3]{x}$

$x^{\frac{1}{n}}$ is another way of writing $\sqrt[n]{x}$.

This alternative notation is often used on calculators. Your calculator may have a function $x^{\frac{1}{y}}$. If so, it can be used to find nth roots as follows.

To find $\sqrt[10]{2}$ or $2^{\frac{1}{10}}$ press $\boxed{2}$ $\boxed{x^{\frac{1}{y}}}$ $\boxed{10}$ $\boxed{=}$ The answer should be 1.071773463.

Activity 5 *n*th root

Find $\sqrt[10]{100}$ to 3 s.f. without using the $x^{\frac{1}{y}}$ or $\sqrt[x]{\ }$ function on your calculator. You **may** use the x^y function.

(Try simpler examples like $\sqrt{100}$ or $\sqrt[3]{100}$ first if you're not sure how to do this.)

Repeat for $\sqrt[100]{1\ 000\ 000\ 000}$ and $\sqrt[10]{0.1}$.

The problem at the beginning of this chapter is an example of 'exponential' growth. This will be dealt with in more detail in Chapter 11 but for the present it can be defined as a growth whereby the number of bacteria present is multiplied by 4 every hour.

Fractional indices are useful in many situations. A good example concerns credit card accounts where any unpaid debt grows exponentially by a certain percentage each month. Credit cards quote a monthly interest rate and the equivalent APR (annual percentage rate).

A typical APR is 29.84%. This means that a debt of £100 at the start of the year becomes one of £129.84 by the end. However, the interest is worked out monthly at a monthly rate of 2.2%. This figure arises because

$$(1.022)^{12} = 1.2984$$

giving an APR of (1.2984–1)100%.

Another way of writing this is to say

$$1.2984^{\frac{1}{12}} = 1.022\,.$$

Example

What is the monthly interest rate if the APR is 34.45%?

Solution

Since $(1.3445)^{\frac{1}{12}} = 1.02498\ldots,$

monthly interest rate = 2.5%.

Example

Joel has an outstanding credit card bill of £162. The APR is 30.23%. He leaves it two months before paying. How much does he have to pay?

Solution

2 months is $\frac{1}{6}$ of a year.

But $(1.3023)^{\frac{1}{6}} = 1.045$

and $1.045 \times 162 = £169.29$

Exercise 9A

1. Calculate these without a calculator:

   (a) $16^{\frac{1}{2}}$ (b) $8^{\frac{1}{3}}$ (c) $81^{\frac{1}{4}}$

   (d) $\left(\frac{1}{4}\right)^{\frac{1}{2}}$ (e) $\left(\frac{16}{625}\right)^{\frac{1}{4}}$ (f) $(-1)^{\frac{1}{3}}$

2. Use a calculator to work these out to 3 s.f.

   (a) $6^{\frac{1}{2}}$ (b) $10^{\frac{1}{3}}$ (c) $56^{\frac{1}{5}}$

   (d) $(0.03)^{\frac{1}{20}}$ (e) $(0.5)^{\frac{1}{2}}$

3. Solve these equations:

   (a) $4^x = 2$ (b) $125^x = 5$

   (c) $100\ 000^x = 10$ (d) $81^x = \frac{1}{3}$

4. The APR on a credit card is 26.08%.

   (a) Jan has an outstanding bill of £365. To how much does this grow in 6 months, assuming no other transactions take place?

   (b) Mark has a bill of £218. What is his bill a month later?

5. Water lilies on a pond grow exponentially so that the area they cover doubles every week. On Sunday they cover 13% of the surface. What percentage do they cover on Monday?

6. An investment policy boasts exponential growth and guarantees to treble your investment, at least, over 20 years. Work out the minimum guaranteed value, to the nearest pound, of

   (a) a £1000 investment after 10 years;

   (b) a £600 investment after 5 years;

   (c) a £2400 investment after 4 years.

9.2 Further problems

Piano tuners also deal with exponential growth. When tuning a piano to 'concert pitch' the first thing to do is to make sure the note A in the middle of the piano is in tune. A properly tuned 'middle A' has a vibrating frequency of 440 Hz (Hz is short for Hertz and means cycles per second).

There are lots of notes called A on the piano. The distance between consecutive As is called an octave. Every octave up, the frequency doubles, as follows :

two As above	1760
one A above	880
middle A	440
one A below	220
two As below	110

Activity 6 Finding the frequency

(a) Write down a formula for the frequency of the note p octaves above middle A.

(b) Use this to find the frequency of the note half an octave above A.

(c) An octave actually consists of 12 notes, so 'half an octave higher' means '6 notes higher'. Find the frequency of the note

(i) 3 notes above middle A

(ii) 5 notes above middle A

(iii) 6 notes below middle A

(iv) 11 notes below middle A

(d) Adapt your formula in (a) to find the frequency of the note n above middle A. Does this formula apply to notes below middle A?

Activity 7

Before reading on, discuss the meaning of these numbers and hence find their numerical value to 3 s.f., explaining your method clearly. Can you find more than one method for some of them?

$$2^{\frac{5}{12}} \quad 5^{\frac{3}{2}} \quad 10^{1.2} \quad 2^{-\frac{11}{12}} \quad 3^{-0.4}$$

There are always at least two ways of thinking about expressions like $3^{\frac{2}{5}}$.

Remembering that $\left(x^m\right)^n = x^{mn}$,

$$3^{\frac{2}{5}} = \left(3^2\right)^{\frac{1}{5}} \quad \text{or} \quad 3^{\frac{2}{5}} = \left(3^{\frac{1}{5}}\right)^2$$

$$= \left(\sqrt[5]{3^2}\right) \qquad\qquad = \left(\sqrt[5]{3}\right)^2$$

$$= \left(\sqrt[5]{9}\right).$$

In general,

$$x^{\frac{p}{q}} = \sqrt[q]{x^p} \quad \text{or} \quad \left(\sqrt[q]{x}\right)^p.$$

Since $\frac{2}{5} = 2 \times \frac{1}{5}$ or $\frac{1}{5} \times 2$,

the order of the root and the power does not matter.

Does this extend to negative indices?

Examples are

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

and $$3^{-\frac{2}{5}} = \frac{1}{3^{\frac{2}{5}}}.$$

Example

Calculate the numerical values of

(a) $2^{\frac{4}{7}}$ (b) $10^{\frac{3}{2}}$ (c) $6^{-\frac{3}{8}}$ (d) $(0.6)^{-\frac{7}{3}}$

Solution

There is always more than one method of working these out. Only one method is shown for each one here. Pay particular regard to the methods in (b) and (d).

(a) $2^{\frac{4}{7}} = \left(2^4\right)^{\frac{1}{7}} = \sqrt[7]{16} = 1.49$ to 3 s.f.

(b) $10^{\frac{3}{2}} = 10^1 \times 10^{\frac{1}{2}} = 10\sqrt{10} = 31.6$ to 3 s.f.

(c) $6^{\frac{3}{8}} = \left(6^{\frac{1}{8}}\right)^3 = (1.251033\ldots)^3 = 1.9579731\ldots$

$$6^{-\frac{3}{8}} = \frac{1}{6^{\frac{3}{8}}} = \frac{1}{1.9579731\ldots} = 0.511 \text{ to 3 s.f.}$$

(d) $(0.6)^{\frac{7}{3}} = (0.6)^{2\frac{1}{3}} = (0.6)^2 \times (0.6)^{\frac{1}{3}}$

$= 0.36 \times \sqrt[3]{0.6} = 0.3036358\ldots$

$(0.6)^{-\frac{7}{3}} = \dfrac{1}{0.3036358\ldots} = 3.29$ to 3 s.f.

(Calculator note : always carry through as many figures as you can until the end of the calculation.)

Exercise 9B

1. Work these out without a calculator.

   (a) $4^{1\frac{1}{2}}$ (b) $27^{\frac{2}{3}}$ (c) $100^{\frac{5}{2}}$

   (d) $1000^{1\frac{1}{3}}$ (e) $16^{\frac{5}{4}}$ (f) $32^{0.4}$

2. Use a calculator to work these out to 3 s.f.

   (a) $120^{\frac{3}{2}}$ (b) $(0.7)^{\frac{5}{3}}$ (c) $5^{3.25}$

   (d) $1000^{\frac{2}{9}}$ (e) $\left(\frac{1}{4}\right)^{\frac{3}{7}}$ (f) $(0.36)^{3.1}$

3. Write these as fractions.

   (a) $16^{-\frac{1}{2}}$ (b) $4^{-1\frac{1}{2}}$ (c) $32^{-0.4}$ (d) $125^{-\frac{4}{3}}$

4. Calculate these to 3 s.f.

   (a) $10^{-\frac{1}{2}}$ (b) $15^{-\frac{1}{3}}$ (c) $(0.2)^{-\frac{5}{2}}$ (d) $(3.5)^{-\frac{4}{3}}$

5. Solve these equations. A calculator is not required.

   (a) $100^x = 100$ (b) $4^x = 32$

   (c) $9^x = \frac{1}{3}$ (d) $8^x = \frac{1}{2}$

   (e) $64^x = 16$ (f) $16^x = \frac{1}{8}$

6. Rewrite these formulas without using fractional or negative indices.

   (e.g. $x^{-1} = \dfrac{1}{x}$; $x^{\frac{3}{2}} = x\sqrt{x}$ or $\sqrt{x^3}$.)

   (a) $5p^{\frac{1}{2}}$ (b) $6q^{-1}$

   (c) $10x^{-\frac{1}{2}}$ (d) $\frac{3}{4}y^{-\frac{1}{2}}$

   (e) $\frac{1}{2}m^{\frac{3}{2}}$ (f) $12t^{-\frac{5}{2}}$

7. The population of a city is expected to grow exponentially and to double in 15 years. By what percentage would you expect the population to have risen after

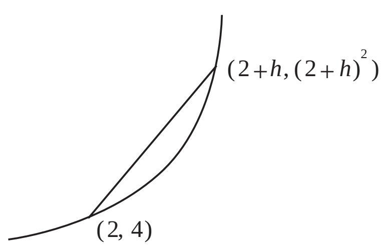
   (a) 4 years; (b) 10 years.

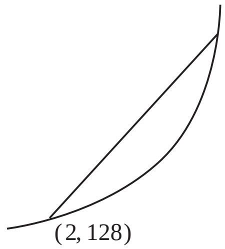
9.3 Binomial expansions

'Binomial' is a word meaning 'two terms', and is used in algebra to mean expressions such as $a+2$ and $2x-y$. (Compare with the word ‘polynomial’).

Binomial expressions were used extensively in Chapter 8; for example the gradient of the chord in this diagram is

$$\frac{(2+h)^2-4}{h}$$





At the heart of this is the binomial $2+h$, raised to a power. For a more complicated curve the gradient might be

$$\frac{(2+h)^7 - 128}{h},$$

which is not so easily simplified because the binomial is raised to a high power. Dealing with expressions like $(2+h)^7$ is the focus of the next sections.

Activity 8 True or False?

Here are six statements about binomials. Which of these are true, and which false?

(a) $2(a+b) = 2a + 2b$ (b) $(a+b) \div 2 = (a \div 2) + (b \div 2)$

(c) $(a+b)^2 = a^2 + b^2$ (d) $(a+b)^3 = a^3 + b^3$

(e) $\sqrt{(a+b)} = \sqrt{a} + \sqrt{b}$ (f) $2^{(a+b)} = 2^a + 2^b$

Writing $(a+b)^2$ for $a^2 + b^2$ is a common mistake (though hopefully one you no longer make!). During Chapter 8 you should have got used to handling expansions like

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

These are simple examples of **binominal expansions**.

Example

Expand $(1+x)^4$

Solution

$$
\begin{aligned}
(1+x)^4 &= (1+x)^2(1+x)^2 \\
&= (1+2x+x^2)(1+2x+x^2) \\
&= 1(1+2x+x^2) + 2x(1+2x+x^2) + x^2(1+2x+x^2) \\
&= 1 + 2x + x^2 + 2x + 4x^2 + 2x^3 + x^2 + 2x^3 + x^4 \\
&= 1 + 4x + 6x^2 + 4x^3 + x^4
\end{aligned}
$$

Exercise 9C

1. Write out the expansion of these :

   (a) $(1+x)^2$ (b) $(1+x)^3$

   (c) $(1+x)^4$ (d) $(1+x)^5$

2. Repeat for these :

   (a) $(a+x)^2$ (b) $(a+x)^3$

   (c) $(a+x)^4$ (d) $(a+x)^5$

3. Write out the expansions of

   (a) $(3-2x)^2$ (b) $(2+5p)^3$ (c) $(\frac{1}{2}m-5)^4$

9.4 Binomial coefficients

You will be thankful that Exercise 9C was neither lengthy nor involved powers higher than the 5th power. As the index rises, so the expansions become more tedious. Expanding $(1+x)^{50}$ by this method, for example, would be somewhat daunting.

Activity 9

(a) Using your answers to Question 1 of Exercise 9C, complete this table :

Coefficients of

	1	x	x^2	x^3	x^4	x^5	x^6	x^7
$(1+x)^1$	1	1						
$(1+x)^2$	1	2	1					
$(1+x)^3$	1	3	3	1				
$(1+x)^4$								
$(1+x)^5$								

(b) What are the coefficients in the 6th and 7th rows?

(c) Write out the expansions of $(1+x)^6$ and $(1+x)^7$.

(d) How does the table above correspond to the expansions of $(a+x)^2, (a+x)^3$ etc.? What do all the terms in the expansion of, say $(a+x)^4$ have in common?

(e) Write out the expansions of $(a+x)^6$ and $(a+x)^7$.

(f) What is the **sum** of the numbers in the nth row?

The numbers in the table are called the **binomial coefficients**. It is evident that they obey a pattern, but finding the numbers in a particular row depends on knowing the numbers in the row before. To expand $(1+x)^{50}$ you would need to know all the coefficients up to and including the 49th row first.

To get a formula for the binomial coefficients it is important to see how they arise. Look more closely at the expression $(a+x)^5$. Remember that it is short for

$$(a+x)(a+x)(a+x)(a+x)(a+x)$$

and that the expansion is

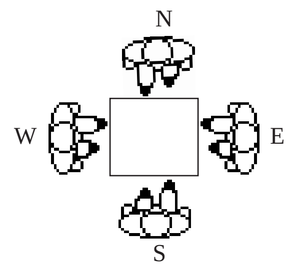
$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5.$$

In each term the two index numbers add up to 5. Moreover, the sum of the coefficients is 32. These two facts reflect what is going on here : each term is the product of a mixture of 'a's and 'x's, one letter from each of the 5 brackets. Each bracket can supply one of two letters, so the number of different combinations of 'a's and 'x's is $2\times2\times2\times2\times2$.

Only one such combination is all 'a's, hence the expansion starts with a single a^5 term. Five of the 32 possibilities give a^4x, 10 of them a^3x^2, and so on. It is the general method of finding the binomial coefficients 1, 5, 10, 10, 5, 1 that is being sought. That is the aim of the next section.

Activity 10 Bridge problem

Bridge is one of the best of all card games. In particular, it is one in which the skills of the player can overcome the luck of the deal. Here is a situation often faced by bridge players.



Suppose you are sitting at S (south). The opposing players are sitting either side of you at W (west) and E (east). You know that between them they have four cards in the heart suit. It is important to know the relative likelihood of various distributions of these four cards between W and E.

Each of the four cards can be in one of two places. Hence there are $2^4=16$ different arrangements.

(a) How many of these arrangements involve all four hearts being with W and none with E?

(b) How many of them involve E having only **one** heart?

(c) Complete the table below.

W	E	No. of ways
4	0	
3	1	
2	2	
1	3	
0	4	
	Total	16

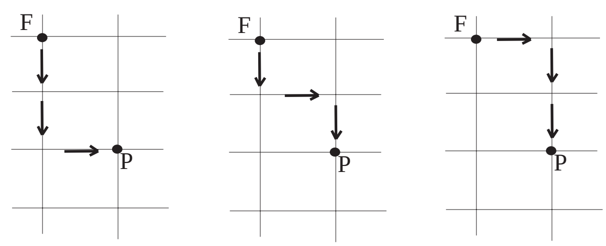
(d) Bridge players sometimes claim that in these circumstances a '1-3 split' (either way) is more likely than an even split. Is this true? Explain.

(e) Extend this to the case where S knows that W and E have 5 hearts between them.

You should by now have realised that it could be dangerous to play bridge with a mathematician!

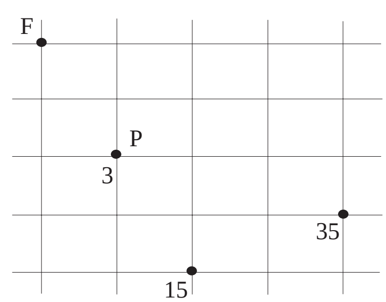
Activity 11 Streets of Manhattan

Many US cities have a street layout based on a rectangular pattern. On Manhattan Island, for instance, more than 200 streets run across the island and ten avenues run at right angles to them.



F represents a fire station. A fire is reported at junction P and the fire services need to take the shortest route from F to P. In fact, there are three equally short routes, as the three diagrams show.

Copy the street plan shown opposite and, by each crossroad, write the number of shortest routes there are from F. Some have been written in to help you.



See if you can discover the link between this problem and the binomial coefficients.

Activities 10 and 11 give the same pattern as that of the binomial coefficients. This is because all three situations can be reduced essentially to the same thing. Here are specific questions and solutions from all three.

Bridge problem

How many ways can 5 cards be divided so that 3 are with E and 2 with W?

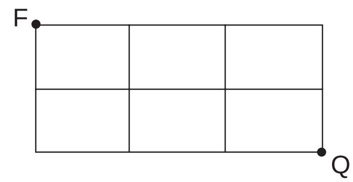
Solution

Imagine all five cards in a row. Each can be marked either E or W. There must be 3 Es and 2 Ws in any order. The answer to the question is, therefore, the number of different combinations possible with 3 Es and 2 Ws. As the list opposite shows, that number is 10.

EEEWW	EWWEE
EEWEW	WEEEW
EEWWE	WEEWE
EWEWE	WEWEE
EWEEW	WWEEE

Streets problem

In this plan, how many shortest routes are there from F to Q?



Solution

F and Q are 5 blocks away. To get to Q you must go across (A) 3 and down (D) 2, in any order. The number of such routes is therefore the number of different combinations of 3 As and 2 Ds. The answer is 10.

AAADD	ADDAA
AADAD	DAAAD
AADDA	DADAA
ADADA	DAADA
ADAAD	DDAAA

Binomial coefficients

In the expansion of $(a+x)^5$, what is the coefficient of the term a^3x^2?

Solution

$(a+x)^5$ is short for $(a+x)(a+x)(a+x)(a+x)(a+x)$. To get a^3x^2 three brackets must supply an 'a' and two of them an 'x'. The coefficient will be the number of ways this can be done. A list of the possible ways is shown on the right.

aaaxx	*axxaa*
aaxax	*xaaax*
aaxxa	*xaaxa*
axaax	*xaxaa*
axaxa	*xxaaa*

It should now be clear why three apparently unrelated problems give identical answers. The number of ways of arranging 3 As and 2 Ds is denoted

$$\binom{5}{3} \quad \text{or} \quad {}^5C_3.$$

The 'C' stands for COMBINATION.

To take another example.

$$\binom{7}{2} \quad \text{or} \quad {}^7C_2$$

means the number of different combinations of 7 objects, 2 of one type and 5 of another.

Exercise 9D

1. From what you have done in this section, write down the values of

   (a) $\binom{4}{2}$ (b) $\binom{5}{1}$ (c) $\binom{5}{2}$ (d) $\binom{6}{3}$ (e) $\binom{3}{1}$

2. How many different ways are there of arranging the letters of the words BOB and ANNA? Express your answers in terms of combinations,

   i.e $\binom{p}{q}$

*9.5 Factorials

Activity 12 Combinations of letters

(a) In the 8-letter word DOMINATE all the letters are different. Including the one given, how many arrangements are there of these letters.

(b) The word NOMINATE also has 8 letters, but two of them are the same. How many arrangements are there now?

(c) The word ADDITION has eight letters with two pairs of identical ones. How many different arrangements are there?

(d) How many arrangements are there of the word CALCULUS?

(e) The word DIVISION has one letter that appears three times. How many arrangements of these eight letters are there?

(f) How many arrangements are there of the words COCOONED and ASSESSES?

(g) Write down a general rule that finds the number of arrangements of the letters of an 8-letter word.

In general, the number of **arrangements** of an n-letter word with one letter repeated p times is given by the formula

$$\frac{n!}{p!}$$

$n!$ read as 'n factorial', is short for the product of all the natural numbers up to and including n; so for example

$$7! = 7\times6\times5\times4\times3\times2\times1 = 5040.$$

The number of different arrangements of letters in the word **SIMILAR** is therefore

$$\frac{7!}{2!} = \frac{7\times6\times5\times4\times3\times2\times1}{2\times1} = 2520.$$

This rule can be extended to cover the repetition of more than one letter. The number of arrangements of letters in the word **SENSES** is

$$\frac{6!}{3!2!}=\frac{6\times5\times4\times3\times2\times1}{3\times2\times1\times2\times1}=60.$$

This automatically provides a way of finding binomial coefficients.

$\binom{5}{3}$ is the number of arrangements of the letters AAADD

so $$\binom{5}{3}=\frac{5!}{2!3!}=\frac{5\times4\times3\times2\times1}{2\times1\times3\times2\times1}=\frac{5\times4}{2\times1}=10.$$

Similarly the binomial coefficient

$$\binom{10}{7}=\frac{10!}{3!7!}=\frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{(3\times2\times1)\times(7\times6\times5\times4\times3\times2\times1)}$$

$$=\frac{10\times9\times8}{3\times2\times1}=120.$$

Note also that

$$\binom{n}{n-r}=\frac{n!}{\left(n-(n-r)\right)!\,(n-r)!}=\frac{n!}{r!(n-r)!}=\binom{n}{r}.$$

Thus the general formula for binomial coefficients is given by

$$\binom{n}{r}=\frac{n!}{(n-r)!r!}$$

Your calculator probably has a factorial function on it. It may also have a function for working out binomial coefficients; consult your manual if you are not certain. In any case, it is often possible to work them out without a calculator; though the formula looks difficult there is always a good deal of cancelling that can be done, as the examples so far have shown.

Example

How many ways are there of selecting a tennis team of 5 from a squad of 9 players?

Solution

Imagine all 9 players in a line, 5 of them are in the team : label them T; 4 are not : label them N. The answer to the question is the number of arrangements of 5 Ts and 4 Ns, since each different arrangement gives a different team.

$$\binom{9}{5} = \frac{9!}{4!5!} = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{(4\times3\times2\times1)(5\times4\times3\times2\times1)} = 126$$

Activity 13

Compose a question to which the answer is $\binom{5}{5}$. Why does this mean that 0! = 1?

*Exercise 9E

1. How many 'words' can be made from
   (a) 6 As and 3 Xs;
   (b) 7 Us and 5 Ws;
   (c) 10 Ys and 13 Zs?
2. How many different ways are there of arranging the letters in these names?
   SADIA
   WILLIAM
   BARBARA
   CHRISTOPHER
3. A committee of four is to be selected from a club of 25 people. How many different possibilities are there?
4. A florist puts ten pot plants in the window. Six of them are red, four green. How many different arrangements of the colours can be made?
5. Eleven hockey players are to be selected from a squad of 16. How many possible selections are there?
6. 12 people are to be split up into
   group A - four people;
   group B - six people;
   group C - two people.

9.6 Binomial theorem

First, there are some results concerning binomial coefficients which will be very useful later.

Activity 14

(a) Show that $\binom{n}{0} = 1, \binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$.

(b) Write down corresponding formulas for $\binom{n}{3}$ and $\binom{n}{4}$.

(c) What is $\binom{n}{r}$?

Activity 15

$$\binom{21}{12} = 293\ 930 \text{ and } \binom{21}{13} = 203\ 490$$

Without using a calculator , write down the values of

$$\binom{21}{9} \text{ and } \binom{22}{13}$$

What properties of the binomial coefficients have you used? Express this symbolically.

The expansion of $(1+x)^5$ can now be written down as follows :

$$1+\binom{5}{1}x+\binom{5}{2}x^2+\binom{5}{3}x^3+\binom{5}{4}x^4+x^5$$

$$=1+5x+10x^2+10x^3+5x^4+x^5.$$

Similarly the expansion of $(a+x)^8$, where a is any number, can be found thus :

$$a^8+\binom{8}{1}a^7x+\binom{8}{2}a^6x^2+\binom{8}{3}a^5x^3+\binom{8}{4}a^4x^4+\binom{8}{5}a^3x^5$$
$$+\binom{8}{6}a^2x^6+\binom{8}{7}ax^7+x^8$$
$$=a^8+8a^7x+28a^6x^2+56a^5x^3+70a^4x^4+56a^3x^8$$
$$+28a^2x^6+8ax^7+x^8$$

The general result is known as the **binomial theorem** for a positive integer index.

For $a, x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$(a+x)^n = a^n+\binom{n}{1}a^{n-1}x+\binom{n}{2}a^{n-2}x^2+\ldots$$
$$\ldots+\binom{n}{r}a^{n-r}x^r+\ldots+x^n$$

Note the way this result is set out. Since n is unknown it is impossible to show the whole expansion. So the formula shows how the expansion starts and finishes, and gives a formula for the general term as

$$\binom{n}{r}a^{n-r}x^r.$$

One use of the binomial theorem is to give rough approximations to numbers like $(1.01)^8$. You can write

$$1.01 = 1 + 0.01$$

so that

$$(1.01)^8 = 1 + \binom{8}{1}(0.01) + \binom{8}{2}(0.01)^2 + \binom{8}{3}(0.01)^3 + \dots$$

$$= 1 + 8 \times 0.01 + 28 \times 0.0001 + 56 \times 0.000001 + \dots$$

If 3 s.f. accuracy is required, clearly only the first three terms are needed at most:

$$(1.01)^8 = 1.0828\ldots = 1.08 \text{ to 3 s.f.}$$

The first four terms give an accuracy of at least 6 s.f.

$$(1.01)^8 = 1 + 0.08 + 0.0028 + 0.000056$$
$$= 1.08286 \text{ to 6 s.f.}$$

And all this without a calculator!

Example

Expand $(2x + 3y)^4$.

Solution

$$(2x+3y)^4 = (2x)^4 + \binom{4}{1}(2x)^3(3y) + \binom{4}{2}(2x)^2(3y)^2 + \binom{4}{3}(2x)(3y)^3 + (3y)^4$$

$$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$$

Example

Expand $(3 - \frac{1}{2}p)^5$

Solution

$$(3-\tfrac{1}{2}p)^5 = 3^5 + \binom{5}{1}3^4(-\tfrac{1}{2}p) + \binom{5}{2}3^3(-\tfrac{1}{2}p)^2 + \binom{5}{3}3^2(-\tfrac{1}{2}p)^3 + \binom{5}{4}3(-\tfrac{1}{2}p)^4 + (-\tfrac{1}{2}p)^5$$

$$= 243 - \frac{405}{2}p + \frac{135}{2}p^2 - \frac{45}{4}p^3 + \frac{15}{16}p^4 - \frac{1}{32}p^5.$$

Example

Find the coefficient of x^5 in the expansion of $(4-3x)^9$.

Solution

The x^5term must be $\binom{9}{5}4^4(-3x)^5$

$$= 126 \times 256 \times (-243)x^5$$

$$= -7\ 838\ 208x^5$$

So the coefficient of x^5 is $-7\ 838\ 208$.

Exercise 9F

1. Expand in full :

   (a) $(1+2x)^4$ (b) $(5-2p)^3$

   (c) $(6-\frac{1}{2}a)^6$ (d) $(2m+3n)^5$

   (e) $(2-\frac{1}{2}r)^4$

2. Find the coefficients of

   (a) x^3 in $(2+x)^{10}$ (b) a^5 in $(5-2a)^7$

   (c) q^6 in $(2p+5q)^9$ (d) k^4 in $(3k-2l)^{12}$

   (e) n^7 in $(5-\frac{5}{6}n)^{10}$

3. Find these to 3 s.f. without using a calculator.

   (a) $(1.01)^5$ (b) $(1.02)^{10}$

   (c) $(0.99)^7$ (d) $(2.01)^3$

   (e) $(99.5)^4$ (f) 11^6

9.7 Miscellaneous Exercises

1. Write these as fractions

   (a) $81^{-\frac{1}{4}}$ (b) $\left(\frac{9}{4}\right)^{\frac{1}{2}}$ (c) $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ (d) $16^{-\frac{3}{4}}$

2. Solve these equations. A calculator is not needed.

   (a) $16^x = 2$ (b) $100^x = 0.1$ (c) $8^x = \frac{1}{4}$

   (d) $25^{2x+1} = 0.04$

3. Given that $3^{2.524} = 16$, solve these equations to 3 d.p.

   (a) $3^x = 4$ (b) $3^x = \frac{1}{2}$ (c) $3^x = \frac{1}{256}$ (d) $9^x = 32$

4. A class of 23 students selects a Year Council representative, four duty team members and two quiz team members. How many ways are there of doing this if

   (a) no-one is allowed to fill more than one of these roles;

   (b) there is no such restriction?

5. Use the binomial theorem to expand:

   (a) $(1+x)^5$ (b) $\left(3+\frac{x}{2}\right)^4$ (c) $(2-3p)^6$

6. Find the coefficient of

   (a) x^6 in $\left(x+x^2\right)^4$ (b) p^3 in $\left(p^2-\frac{3}{p}\right)^6$

   (c) n^4 in $(2+n)^3(1+2n)^4$

   (d) y^6 in $(5-2y)^4(y-1)^5$

7. Determine the coefficient of x^3 in the binomial expansion of $(1-2x)^7$. (AEB)