21 MATHEMATICAL MODELLING

Objectives

After studying this chapter you should

- understand how mathematical models are formulated, solved and interpreted;
- appreciate the power and limitations of mathematics to solve practical problems.

21.0 Introduction

Throughout this text, you will have seen many techniques applied to a variety of problems. This has often involved setting up a mathematical model to describe the problem, and this concept was introduced in Section 1.2.

You now have many more mathematical techniques available to solve problems, and in this last chapter you will need to use many of them!

For each case study, the following procedure will be observed:

- specify the real problem
- set up a mathematical model
- determine the mathematical problem
- solve the mathematical problem
- interpret the solution
- use the solution or improve the model.

21.1 Stock control

Specify the real problem

In this case study you will consider a problem faced by management in industry. Every large organisation needs a policy for controlling its stock levels. The simple model for stock control considered here is fundamental to many large firms. Here is a particular example.

(i) Data on costs associated with stock handling are as follows. The **ordering cost** (cost of administrative processing) is estimated to be £25 per order. The annual **holding cost** of an item in store (mainly interest charges being paid on borrowed capital, or loss of interest which could be made on that capital if it were lent elsewhere) is assessed at 17% of the goods' price.

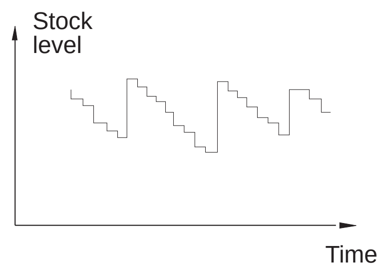
(ii) The ordering of stock is at the disposal of the person managing the stockholding, who must take two decisions:

- how much to order
- when to order.

(iii) The problem is to find the cheapest decisions.

Set up a model

If there were no systematic attempt to 'manage' the stock-keeping then ,for any particular item held in store, the graph of stock level against time would look like the graph opposite.



Each vertical jump upwards represents a re-stocking, and the downward 'staircases' represent usage of stock.

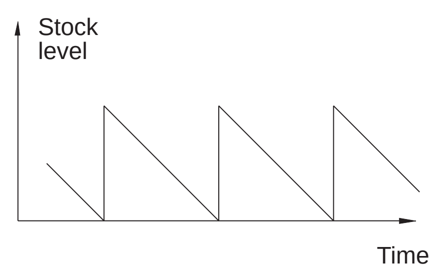
You can simplify and idealize the problem by making the following assumptions:

(a) that orders are placed at regular intervals, each order being for the same quantity;

(b) that the order is placed just as stocks run out (since holding cost increases with amount of stock held) and is filled instantaneously;

(c) that the rate of usage is constant and that the stock level can vary continuously.

These three assumptions may be summarized by their effect on the 'realistic' graph of stock level against time shown above. The 'idealized' graph is shown opposite. There is now a regular series of 'teeth' of height equal to the reorder quantity.



Formulate the mathematical problem

Introduce variables

Let the reorder quantity be q items and the rate of usage u items per year. The total annual cost (annual ordering cost plus annual holding cost) for the stock is denoted by £y, and the (constant) price of each individual item by £P.

Relate the variables

Total annual cost has been defined as annual ordering cost plus annual holding cost. From the data given above

ordering cost per order $= £25$, and

annual holding cost per item $=$ 17% of its cost price

$= £(0.17P)$.

Rate of usage is u items per year, and there are q items in each order. The annual number of orders placed is therefore u/q, so that

$$\text{annual ordering cost} = £\left\{25 \times \frac{u}{q}\right\}$$

The 'idealized' graph shows that stock level drops from q to 0 at a uniform rate between successive orders. It follows that the average stock level is $\frac{1}{2}q$ items. Each item held costs $£(0.17P)$ per year. Hence

$$\text{annual holding cost for entire stock} = £\left(0.17P \times \frac{1}{2}q\right)$$

The total annual cost, £ y, can be expressed in terms of P, q and u:

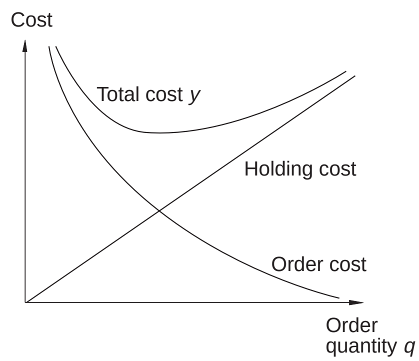
$$y = \frac{25u}{q} + \frac{0.17Pq}{2}$$

Mathematical problem

In the boxed formula above, P and u are assumed constant. The mathematical problem is to find a positive value of q which minimises y if such exists.

Solve the mathematical problem

The graph of the function y has a minimum for a positive value of q, as illustrated opposite.



To find the stationary point, you differentiate y with respect to q, remembering that P and u are constants. This gives

$$\frac{dy}{dq} = -\frac{25u}{q^2} + \frac{0.17P}{2}$$

The value of q for which $\frac{dy}{dq}$ is zero is given by the equation

$$-\frac{25u}{q^2}+\frac{0.17P}{2}=0 \; .$$

This gives the solution for q as

$$\boxed{q=\sqrt{\frac{2\times 25u}{0.17P}}}$$

Interpret the solution

The optimal order quantity is given as

$$\sqrt{\frac{2\times 25\times \text{annual usage}}{0.17\times \text{ cost price per item}}}$$

More generally, the optimal order quantity is

$$\sqrt{\frac{2\times \text{ordering cost}\times \text{annual usage}}{\text{holding cost per item}}}$$

so that

(a) higher holding cost (higher price) means a smaller optimal order quantity;

(b) higher annual usage implies a larger optimal order quantity, but (for example) it takes a four-fold increase in usage to double the optimal order quantity.

The two qualitative predictions agree with what one would expect.

Improve the model

The model contains some unrealistic features for which allowance should be made.

(a) It assumes that orders are placed at the moment stock runs out, and that they are instantaneously satisfied. There is in fact a gap, called the lead time, between the placing of an order and goods' arrival.

(b) Stock may be allowed to run out before restocking (the associated run-out cost can be assessed and traded off against the reduced holding cost).

(c) Demand is not steady.

(d) Prices vary with time.

Incorporating these features would make the mathematics vastly more complicated, and it is often a balance between realism and complex mathematics that has to be sought.

Activity 1 Using the model

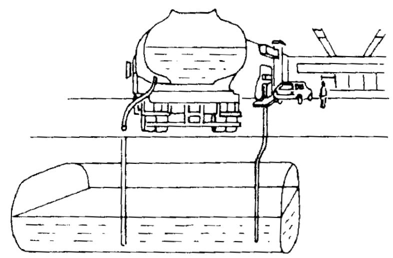
Find the optimal order quantity when $P = 2$ and

(a) $u = 50000$ (b) $u = 5000$ (c) $u = 500$.

In each case, also find the number of orders made per year.

21.2 Dipstick

Specify the real problem

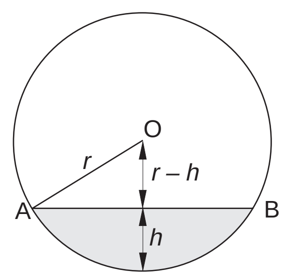


Petrol stations very rarely run out of petrol. This is due partly to efficient deliveries but also to precise stock control.

Each type of petrol (4 star, unleaded, diesel) is stored in an underground tank and the amount left in each tank is carefully monitored using some form of dipstick.

Our problem is to design the calibration on a dipstick.

Set up a model



It is easy to measure the **height**, say h, left in the tank. However, the volume will be proportional to the **cross-sectional area** - not the height. Suppose the cross-section is a circle (it is in fact elliptical). You will need to find the relationship between the shaded area , A, and the height, h, and so provide a ready reckoner to convert height to area.

Formulate the mathematical problem

For simplicity, take $r = 1$ m. For values of h from 0 to 1.0, you need to find the angle, θ, and hence the cross-sectional area of oil.

Solve the mathematical problem

You can see that, with θ in degrees,

$$\cos\theta = \frac{r-h}{r} = 1-h \quad (\text{since } r = 1)$$

The area of the sector OAB is given by

$$\left(\frac{2\theta}{360}\right)\times\pi\times 1^2=\frac{\pi\theta}{180}$$

whereas the area of the triangle OAB is given by

$$2\times\left(\frac{1}{2}\times(1-h)\times 1\sin\theta\right)=(1-h)\sin\theta\,.$$

Hence the cross-sectional area of oil is given by

$$\frac{\pi\theta}{180}-(1-h)\sin\theta$$

The whole cross-sectional area of the tank is π, so the fraction A' of the whole area which corresponds to the oil is given by

$$A'=\frac{\theta}{180}-\frac{(1-h)}{\pi}\sin\theta$$

Interpret the solution

You can use this formula to complete a ready reckoner.

Activity 2

Complete the task opposite by using the equation

$$\cos\theta=1-h$$

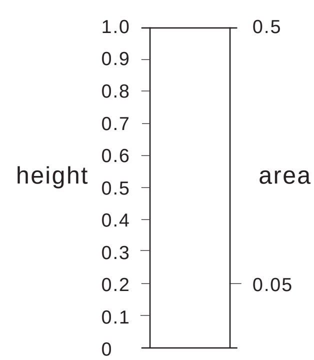
to determine θ for given h, and the formula above to find A'.

Plot a graph of A' on the vertical axis against height h on the horizontal cross. Use your graph to estimate the height that corresponds to an area fraction of

(a) 0.05 (b) 0.10.

h	$\theta°$	A'
0	0	o
0.1	25.84	0.019
0.2	36.87	0.052
...	...	...
...	...	...
1.0	90	0.500

You are now in a position to design the dipstick. Part of it is shown opposite. The height on the left hand side is linear, whereas the area fraction is not. Since $h=0.2$ corresponds to $A'=0.05$ approximately, you can see that $A'=0.05$ is indicated at the same level as $h=0.2$.



Activity 3

Use your graph to find the values of h which correspond to $A' = 0.1, 0.15, 0.2, \ldots, 0.45$, and use these values to complete the dipstick calibration.

21.3 Dicey games

Specify the real problem

You have to organise a game of chance at the school fete in which contestants pay a 10p entry fee to roll three dice. The total score is recorded and there are money prizes for high scores. The problem is to decide whether these prizes will give sufficient incentive for people to enter the game, and sufficiently low for you, the organiser to make a profit!

Throw 3 dice

Entry Fee : 10p

Win

£1 for 18

50p for 17, 16, 15

20p for 14, 13, 12

Set up a model

The model here is straightforward - you associate a probability of $\frac{1}{6}$ with obtaining any number 1, 2, ..., 6 with **one** roll of a dice.

Formulate the mathematical problem

The mathematical problem is to determine the expected winnings/ losses for a contestant.

Solve the mathematical problem

To find the expected winnings, you must find the probabilities of obtaining a total score of 18, 17, ..., 12 since these are the scores which attract a prize.

What is the probability of obtaining 18?

Activity 4

Complete the table below which gives the probabilities of a total sum of 3, 4, ... 18.

3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\frac{1}{216}$													$\frac{6}{216}$	$\frac{3}{216}$	$\frac{1}{216}$

(Check that your sum of the probabilities is 1.)

With these values for the probabilities you can now determine the expected winnings/losses for a contestant. They will win

£1 with a probability of $\frac{1}{216}$

50p with a probability of $\frac{19}{216}$

20p with a probability of $\frac{61}{216}$

Check from your table in Activity 4 the values $\frac{19}{216}$, $\frac{61}{216}$ used above.

So the contestant has an expected profit/loss, in pence, of

$$100 \times \frac{1}{216} + 50 \times \frac{19}{216} + 20 \times \frac{61}{216} - 10$$

$$= \frac{2270}{216} - 10$$

$$= \frac{110}{216}$$

$$\approx \tfrac{1}{2}\text{p}.$$

Interpret the solution

Since, on average, a participant will win $\frac{1}{2}$p, if 100 people play the game during the fete, you will expect to lose about

$$100 \times \tfrac{1}{2}\text{p} = 50\text{p}$$

which will not be of much help to the school funds!

Improve the model

Its not really the model that needs improving, but the money prizes on offer.

Activity 5

Suggest new prizes so that on average a contestant will lose about 5p per game.

21.4 Postscript

The three case studies in this chapter should be sufficient to show you that applying maths can be very useful, but this is not the whole story. The real world is a very complex system and it is up to mathematicians to use their knowledge and skills to help the human race to find optimum solutions to problems.

The ability to perform and understand mathematical techniques and concepts is something to be proud of and to nurture throughout your life. It can at times be frustrating when, for example, integrals do not work out – but the delight and joy in completing a mathematical task correctly is an experience that no one can take away!