18 EVEN MORE CALCULUS

Objectives

After studying this chapter you should

- be able to differentiate and integrate basic trigonometric functions;
- understand how to calculate rates of change;
- be able to integrate using substitutions;
- be able to integrate by parts;
- be able to formulate and solve simple first order differential equations.

18.0 Introduction

In the earlier chapters 8, 11, 12 and 14 you met the concept of differentiation and integration. You should be familiar with the differentiation and integration of functions such as x^n, e^x and ln x. In this chapter the range of functions is expanded to include the differentiation and integration of trigonometric functions and two important methods of integration, namely substitution and by parts, are dealt with. Finally, there is an introduction to the important topic of differential equations.

18.1 Derivatives of trig functions

Intuitive ideas

The derivative of a general function f(x) is given by

$$f'(x) = \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} \right\}$$

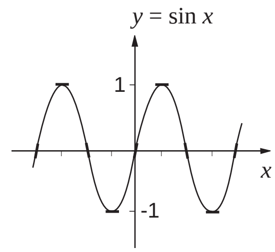
You are going to obtain the derivatives of sin x, cos x, and tan x.

Remember that the derivative of a function at a given point is given by the **gradient** of that function at the given point.

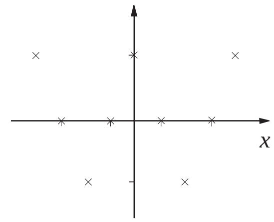
Suppose you look at a sketch of the graph sin x and sketch in the tangents for values of x of:

$$-\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

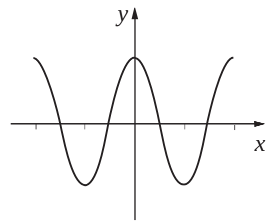
This is shown opposite.



These tangent values when plotted against x will look like the sketch opposite.



This suggests that the graph of the function for the tangent may be something like the sketch opposite.



This sketch suggests that the derivative of sin x might be cos x, but this illustration is certainly not a proof.

Activity 1

By considering the graph of cos x, with tangents drawn at key points, show that the derivative of cos x might be – sin x.

Using a calculator or computer

The derivative of sin x is given by

$$\frac{d}{dx}(\sin x) = \lim_{h\to 0}\left\{\frac{\sin(x+h) - \sin x}{h}\right\}$$

Using a calculator or computer you can try some specific examples - using a set value for x and making h small.

Why is it not possible to take the actual value when $h = 0$?

For instance, setting $x = 0.2$ and $h = 0.00004$,

$$\begin{aligned}\frac{\sin(0.20004) - \sin(0.2)}{0.00004} &= \frac{0.198708533 - 0.198669331}{0.00004} \\ &= \frac{0.000039202}{0.00004} \\ &= 0.98005 \\ &= \cos(0.200083427)\end{aligned}$$

So, approximately

$$\frac{\sin(0.20004) - \sin(0.2)}{0.00004} \approx \cos(0.2)$$

Activity 2

Either calculate

$$\frac{\sin(x+h) - \sin x}{h}$$

for other values of x and very small values of h;

or write a computer programme to evaluate

$$\frac{\sin(x+h) - \sin x}{h}$$

for various values of x and very small values of h.

In each case show that

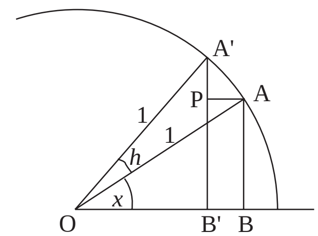
$$\frac{\sin(x+h) - \sin x}{h} \approx \cos x$$

Geometric proof

It is also possible to use the unit circle to demonstrate that, for example,

$$\frac{d}{dx}(\sin x) = \cos x$$

In the diagram, shown opposite, h is small, and



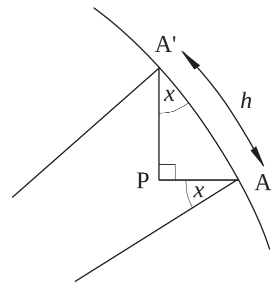
$$A'B' = \sin(x+h)$$

$$AB = \sin x$$

$$A'A = h \quad (\text{radius} \times \text{angle})$$

So $\quad \sin(x+h) - \sin x = A'P$

Now assume that when h is very small the arc $A'A$ is approximately straight.



So for the 'triangle' $AA'P$,

$$\frac{\sin(x+h)-\sin x}{h} \approx \frac{A'P}{A'A} = \cos x$$

So, approximately, if h is very small

$$\frac{\sin(x+h)-\sin x}{h} \approx \cos x$$

Now, taking the limit as $h \to 0$, gives

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

Activity 3

By considering the same unit circle diagram, show that

$$\frac{\cos(x+h)-\cos x}{h} \approx -\sin x$$

and, taking the limit as $h \to 0$,

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

To find the derivative of tan x, formally you can use the quotient rule,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2} \quad \left(u' = \frac{du}{dx},\ v' = \frac{dv}{dx}\right),$$

giving

$$\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\frac{d}{dx}(\sin x)\times\cos x - \sin x\times\frac{d}{dx}(\cos x)}{(\cos x)^2} \\
&= \frac{\cos x\times\cos x - \sin x\times(-\sin x)}{(\cos x)^2} \\
&= \frac{(\cos x)^2+(\sin x)^2}{(\cos x)^2} \\
&= \frac{1}{(\cos x)^2}
\end{aligned}$$

This gives

$$\boxed{\frac{d}{dx}(\tan x) = (\sec x)^2}$$

Activity 4

Show that

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

So the three crucial results are

$$\frac{d}{dx}(\sin x) = \cos x \quad \text{or} \quad y=\sin x \Rightarrow \frac{dy}{dx}=\cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x \quad \text{or} \quad y=\cos x \Rightarrow \frac{dy}{dx}=-\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \text{or} \quad y=\tan x \Rightarrow \frac{dy}{dx}=\sec^2 x$$

and, inverting these results

$$\boxed{\begin{aligned}
&\int \cos x\,dx = \sin x + c \\
&\int \sin x\,dx = -\cos x + c \\
&\int \sec^2 x\,dx = \tan x + c
\end{aligned}}$$

Why is it that we cannot find $\frac{d}{dx}(\tan x)$ when $x = \pm\frac{1}{2}\pi, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi, \ldots$?

To differentiate $y = \sin kx$, where k is a constant, you can define a new variable z by $z = kx$, so that

$$y = \sin kx = \sin z,$$

giving $$\frac{dy}{dz} = \cos z.$$

But $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} \qquad \text{(function of a function)}$$

and since $$z = kx$$

then $$\frac{dz}{dx} = k$$

So $$\frac{dy}{dx} = (\cos z) \times k$$

giving $$\frac{dy}{dx} = k \cos kx.$$

Exercise 18A

1. Find the derivative of

   (a) $\sin 2x$ (b) $\sin \frac{1}{2}x$

   (c) $\sin 100x$ (d) $\cos 3x$

2. Find $\frac{d}{dx}(\cos kx)$ and $\frac{d}{dx}(\tan kx)$

3. Differentiate

   (a) $\sin^2(2x)$ (b) $\sin 2x \cos 2x$

   (c) $\cos^3\left(\frac{1}{2}x\right)$ (d) $\tan^2(2x)$

4. Find $\frac{d}{dx}(\operatorname{cosec} x)$ and $\frac{d}{dx}(\sec x)$.

   Hence determine

   $$\int \sec x \tan x \, dx.$$

18.2 Rates of change

From your earlier work on differentiation, you will recognise that the rate of change of a variable, say f, is given by $\frac{df}{dt}$, where t denotes time. Often though, f is a function of another variable, say x, so that you need to use the 'function of a function' rule,

$$\boxed{\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}}$$

We will see in the next two examples, how this result is used.

Example

The radius of a circular oil slick is increasing at the rate of 2 m/s. Find the rate at which the area of the slick is increasing when its radius is (a) 10 m, (b) 50 m.

Solution

Now $\quad A = \pi r^2$

giving
$$\begin{aligned}\frac{dA}{dt} &= \frac{d}{dt}\left(\pi r^2\right)\\ &= \frac{d}{dr}\left(\pi r^2\right)\frac{dr}{dt} \qquad \text{(using 'function of a function')}\\ &= 2\pi r\frac{dr}{dt}\\ &= 4\pi r, \quad \text{since } \frac{dr}{dt} = 2\text{ m/s}.\end{aligned}$$

(a) When $r = 10\text{ m}$, $\frac{dA}{dt} = 40\pi \approx 125.7\text{ m}^2/\text{s}$.

(b) When $r = 50\text{ m}$, $\frac{dA}{dt} = 200\pi \approx 628.3\text{ m}^2/\text{s}$.

Example

A spherical balloon is blown up so that its volume increases at a constant rate of 10 cm^3/s. Find the rate of increase of the radius when the volume of the balloon is 1000 cm^3.

Solution

Now $\quad v = \frac{4}{3}\pi r^3$

giving $\quad \frac{dv}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)\frac{dr}{dt}$

$$= 4\pi r^2 \frac{dr}{dt}.$$

But $\quad \frac{dv}{dt} = 10$

giving $\quad 10 = 4\pi r^2 \frac{dr}{dt}.$

Thus $\quad \frac{dr}{dt} = \frac{5}{2\pi r^2}.$

When the volume is 1000 cm the radius can be calculated from

$$1000 = \frac{4}{3}\pi r^3$$

$$\Rightarrow \quad r = \left(\frac{750}{\pi}\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{5}{2\pi}\left(\frac{750}{\pi}\right)^{-\frac{2}{3}} \approx 0.021$$

Exercise 18B

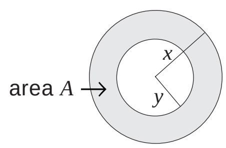
1. The radius of a circular disc is increasing at a constant rate of 0.003 cm/s. Find the rate at which the area is increasing when the radius is 20 cm.
2. A spherical balloon is inflated by gas being pumped at a constant rate of 200 cm^3/s. What is the rate of increase of the surface area of the balloon when its radius is 100 cm?
3. The volume of water in a container is given by xe^{-2x} cm^3 where x is the depth of the water in the container. Find the rate of increase of volume when the depth is 0.2 cm and increasing at a rate of 0.1 cm s^{-1}. Also determine when the rate of increase of volume is stationary.

   (AEB)
4. The area of the region between two concentric circles of radii x and y $(x > y)$ is denoted by A. Given that x is increasing at the rate of 3 ms^{-1}, y is increasing at the rate of 4 ms^{-1}, and when $t = 0, x = 5$ m and $y = 2$ m, find

   (a) the rate of increase of A when $t = 0$;

   (b) the ratio of x to y when A begins to decrease;

   (c) the time at which A is zero.



18.3 Integration by substitution

This method is the equivalent to using a change of variable when differentiating composite functions.

Activity 5

Differentiate

(a) $\left(2x^2-2\right)$ (b) $\left(3x^2+2\right)^5$

(c) $\left(4x^2-7\right)^6$ (d) $\left(4x^2-7\right)^n$

and use the above results to find

(a) $\int x\left(x^2-3\right)^5 dx$ (b) $\int x\left(x^2+2\right)^2 dx$

(c) $\int (3x-1)^5 dx$ (d) $\int x^2\left(x^3-1\right)dx$

(Check your results by differentiation.)

In general, if

$$y=\int f(x)dx$$

then $$\frac{dy}{dx}=f(x).$$

If we let $x=g(u)$ for some function g of a new variable u, then y becomes a function of u and

$$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}$$

$$\Rightarrow \quad \frac{dy}{du}=f(x)\frac{dx}{du}=f(g(u))\frac{dx}{du}$$

$$\Rightarrow \quad y=\int f(g(u))\frac{dx}{du}du$$

Example

Find $\int x(3x-1)^{\frac{1}{2}}\,dx$ using a suitable substitution.

Solution

If $u = 3x - 1$ then $\dfrac{du}{dx} = 3$ and $x = \dfrac{u+1}{3}$. Hence the integral becomes

$$\begin{aligned}
\int x(3x-1)^{\frac{1}{2}} \frac{dx}{du}\,du &= \int \left(\frac{u+1}{3}\right) u^{\frac{1}{2}} \frac{du}{3} \\
&= \tfrac{1}{9}\int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du \\
&= \tfrac{1}{9}\left(\tfrac{2}{5}u^{\frac{5}{2}} + \tfrac{2}{3}u^{\frac{3}{2}}\right) + c \\
&= \tfrac{1}{9}\left(\tfrac{2}{5}(3x-1)^{\frac{5}{2}} + \tfrac{2}{3}(3x-1)^{\frac{3}{2}}\right) + c \\
&= \tfrac{2}{45}(3x-1)^{\frac{5}{2}} + \tfrac{2}{27}(3x-1)^{\frac{3}{2}} + c
\end{aligned}$$

Example

Find $\int_0^1 x\left(3x^2+2\right)^3 dx$

Solution

You can rewrite this integral using the substitution

$$u = \left(3x^2+2\right) \quad \Rightarrow \quad \frac{du}{dx} = 6x$$

The limits on u for the integration are determined from

$$x = 0 \quad \Rightarrow \quad u = 2$$

$$x = 1 \quad \Rightarrow \quad u = 5$$

Hence $\displaystyle\int_0^1 x\left(3x^2+2\right)^3 dx = \int_2^5 x\left(3x^2+2\right)^3 \frac{dx}{du}\,du$

$$= \int_2^5 x\left(3x^2+2\right)^3 \frac{1}{6x}\,du$$

$$= \int_2^5 \frac{1}{6}u^3 du$$

$$= \left[\frac{1}{24}u^4\right]_2^5$$

$$= \frac{1}{24}\left[5^4-2^4\right]$$

$$= \frac{1}{24}(625-16)$$

$$= \frac{609}{24}$$

Exercise 18C

1. Find the following integrals, using the suggested substitution.

   (a) $\displaystyle\int x\left(x^2-3\right)^4 dx$ $\quad\left(u=x^3-3\right)$

   (b) $\displaystyle\int x\sqrt{1-x^2}\,dx$ $\quad\left(u=1-x^2\right)$

   (c) $\displaystyle\int \cos x\sin^4 x\,dx$ $\quad\left(u=\sin x\right)$

   (d) $\displaystyle\int e^x\sqrt{1+e^x}\,dx$ $\quad\left(u=1+e^x\right)$

2. Using the substitution $x=4\sin^2\theta$ or otherwise, show that

$$\int_0^2 \sqrt{x(4-x)}\,dx = \pi$$

3. Using the substitution $x=3\tan\theta$, evaluate

$$\int_0^3 \frac{1}{\left(9+x^2\right)^2}\,dx$$

(AEB)

18.4 Integration by parts

Our second method of integration is a very important one which is particularly useful for a number of different integrals. You can see how it works by first differentiating with respect to x,

$$x \sin x + \cos x$$

Now use your answer to find the integral of $x \cos x$.

What form will the integral of $x \sin x$ take?

Check your answers by differentiating.

Activity 6

By assuming

$$\int x\, e^x dx = (ax + b)e^x + c$$

find the values of the constants a and b so that the result is valid.

(Hint: differentiate the R.H.S.)

By now you should be wondering if there is a more precise method of evaluating integrals of the type above, where the method requires inspired guesswork to start with. In fact, there is a more formal method.

If u and v are two functions of x then from your work on differentiation you know that

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

and integrating each side with respect to x gives

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\Rightarrow \quad \boxed{\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx}$$

Example

Find $\int x \cos x \, dx$

Solution

Here $u = x$ and $\frac{dv}{dx} = \cos x$, so $v = \sin x$. This gives

$$\int x \cos x \, dx = x \sin x - \int \sin x . 1 dx$$

$$= x \sin x + \cos x + c$$

We can use the same method to find the integral in Activity 6.

Example

Find $\int x \, e^x dx$

Solution

Here $u = x$, $\frac{dv}{dx} = e^x$, giving $v = e^x$ and

$$\int x \, e^x dx = x \, e^x - \int e^x . 1 dx$$

$$= x \, e^x - e^x + c$$

$$= e^x (x - 1) + c$$

Activity 7

(a) Given that $\int x \cos x \, dx = x \sin x + \cos x + c$ show that

$$\int x^2 \sin x \, dx = x^2 \cos x + 2x \sin x + 2 \cos x + k$$

(b) Given that $\int x \, e^x dx = x \, e^x - e^x + c$ find

(i) $\int x^2 e^x dx$ (ii) $\int x^2 e^{-x} dx$

One of the difficulties with integration by parts is that of identifying u and $\frac{dv}{dx}$. For example, if

$$\mathrm{I} = \int x \, 1\mathrm{n}\, x \, dx$$

then taking $u = x$, $\frac{dv}{dx} = \ln x$ will not really work as you might not be able to solve for v. But taking

$$u = 1\mathrm{n}\, x, \frac{dv}{dx} = x$$

gives $v = \frac{1}{2}x^2$, and

$$\mathrm{I} = \tfrac{1}{2}x^2 1\mathrm{n}\, x - \int \left(\tfrac{1}{2}x^2\right)\frac{1}{x}\, dx$$

$$= \tfrac{1}{2}x^2 1\mathrm{n}\, x - \int \tfrac{1}{2}x \, dx$$

$$= \tfrac{1}{2}x^2 1\mathrm{n}\, x - \tfrac{1}{4}x^2 + c$$

It is also important to note that some functions which cannot be immediately integrated can be dealt with by writing the integrand as $1\times$(function) since 1 can be integrated. This is illustrated below.

Example

Find $\int 1\mathrm{n}\, x \, dx$

Solution

$$\mathrm{I} = \int 1\mathrm{n}\, x \, dx = \int 1 \times 1\mathrm{n}\, x \, dx$$

So take $u = 1\mathrm{n}\, x$, $\frac{dv}{dx} = 1$ to give $v = x$ and

$$\mathrm{I} = x 1\mathrm{n}\, x - \int x . \frac{1}{x} dx$$

$$= x 1\mathrm{n}\, x - \int 1 dx$$

$$= x 1\mathrm{n}\, x - x + c$$

Finally in this section it should be noted that we can have limits on integration in the usual way, for example

$$\int_0^{\frac{\pi}{2}} x\sin x\,dx = [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1(-\cos x)\,dx$$

$$= (0-0) + \int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= 1$$

Exercise 18D

1. Find $\int x^2 \ln x\,dx$.

2. By writing $\cos^3\theta$ as $\cos^2\theta\cos\theta$ find

$$\int \cos^3\theta\,d\theta$$

and hence show that

$$\int_0^{x} \cos^3\theta\,d\theta = 0.$$

3. Evaluate $\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$.

4. Find $\int x^3 \ln(4x)dx$ (AEB)

5. Use integration by parts twice to find

$$\int x^2\cos 2x\,dx$$

18.5 First order differential equations

You might well wonder why you have studied integration in such depth. The main reason is the crucial role it plays in solving what are known as **differential equations**. Here you will only be dealing with first order differential equations, which can be expressed, in general, in the form

$$\frac{dy}{dx} = f(x, y)$$

when $y = y(x)$ is an unknown function of x, and f is a given function of x and y.

Examples of such equations include

$$\frac{dy}{dx} = xy$$

$$\frac{dy}{dx} = x + y$$

$$\frac{dy}{dx} = \frac{x}{y} \qquad (y \neq 0)$$

but first you will see how they are formed.

As an example of the process, consider the problem of modelling the way in which human populations change.

An English economist, *Thomas Malthus*, in his 1802 article "An Essay on the Principle of Population" formulated the first model which attempted to describe and predict the way in which the magnitude of the human population was changing. In mathematical terms you can summarise his assumptions by introducing the variable $N = N(t)$ to represent the total population, where t is time. In a small time interval, say δt, Malthus argued that both births and deaths are proportional to the population size and the time interval. Hence in time δt, there will be $(\alpha N \delta t)$ births and $(\beta N \delta t)$ deaths where α, β are positive constants, and so the increase in the population size is given by

$$\delta N = \alpha N \delta t - \beta N \delta t = (\alpha - \beta) N \delta t$$

Writing $\alpha - \beta = \gamma$,and dividing by δt gives

$$\frac{\delta N}{\delta t} = \gamma N$$

Taking the limit as $\delta t \to 0$ results in the differential equation

$$\boxed{\frac{dN}{dt} = \gamma N}$$

Added to this is an initial condition; for example, $N = N_0$ at $t = 0$.

This is an example of a 'variable separable' differential equation, since it can be written in the form

$$\frac{1}{N}\frac{dN}{dt} = \gamma$$

and integration with respect to t,

$$\int \frac{1}{N}\frac{dN}{dt}\,dt = \int \gamma\,dt$$

$$\int \frac{1}{N}\,dN = \int \gamma\,dt$$

$$\Rightarrow \quad \ln N = \gamma t + c \quad (c \text{ constant})$$

$$\Rightarrow \quad N = e^{\gamma t + c}$$

$$= e^{\gamma t} e^{c}$$

$$\Rightarrow \quad N = Ke^{\gamma t} \quad \left(K = e^{c}\right)$$

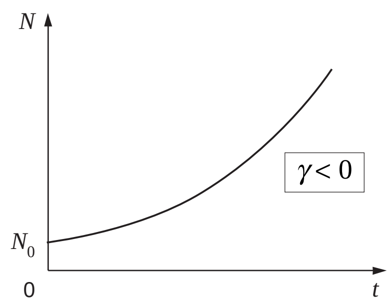
But $\quad N = N_0$ at $t = 0$,

giving $\quad N_0 = Ke^0 = K$.

Hence the complete solution is

$$\boxed{N = N_0 e^{\gamma t}}$$

This solution is illustrated opposite for positive γ.



What does the solution look like when

(a) $\gamma = 0$ (b) γ is negative?

Activity 8

By differentiation check that $N = N_0 e^{\gamma t}$ satisfies the differential equation

$$\frac{dN}{dt} = \gamma N$$

You can see how to use this population model by applying it to the 1790 and 1800 USA population figures shown opposite.

Year	USA Population (in millions)
1790	3.9
1800	5.3

The time $t = 0$ corresponds to 1790, and $t = 1$ to 1800 etc; so

$$N_0 = 3.9$$

The model solution is

$$N(t) = 3.9e^{\gamma t}$$

and γ is determined by the population value at $t = 1$ (1800). This gives

$$5.3 = 3.9e^{\gamma}$$

$$\Rightarrow \quad e^{\gamma} = \frac{5.3}{3.9}$$

$$\Rightarrow \quad \gamma = \ln\left(\frac{5.3}{3.9}\right) \approx 0.03067$$

and $$\boxed{N(t) = 3.9e^{(0.03067)t}}$$

What does this model predict for the population if t becomes large? Is this realistic?

Activity 9

Use the model above to predict the USA population for 1810 $(t = 2)$ to 1930 $(t = 14)$. Compare your prediction with the real values given opposite.

Can you explain any discrepancies in the prediction and real values?

Year	USA Population (in millions)
1810	7.2
1820	9.6
1830	12.9
1840	17.1
1850	23.2
1860	31.4
1870	38.6
1880	50.2
1890	62.9
1900	76.0
1910	92.0
1920	106.5
1930	123.2

The model gives a reasonably good fit for quite some time, but eventually predicts a population growing far too quickly.

Why does the model eventually go wrong?

Let us go back to the original assumption and consider what factors have been neglected. The model as it stands predicts the growth of a population in an ideal situation where there are no limits (e.g. food, land, energy) to continued growth. In 1837, Verhulst proposed an extension to the Malthusian model which took into account the 'crowding' factor. In his model, he assumed that the population change was proportional to

(i) the current population level N

(ii) the ratio of unused population resources to total population resources when it is assumed that a maximum population size of N_∞ can be sustained,

i.e. $$\frac{(N_\infty - N)}{N_\infty}$$

Thus

$$\frac{dN}{dt}=\gamma N\frac{(N_\infty - N)}{N_\infty}$$

i.e. $$\boxed{\frac{dN}{dt}=\gamma N-\frac{\gamma N^2}{N_\infty}}$$

When N is small, γN will be the dominant term on the right hand side, and so we are back with the Malthusian model; but as N becomes large, the $\dfrac{-\gamma N^2}{N_\infty}$ term becomes important.

This is again a first order differential equation, but much more difficult to solve.

*Activity 10

Show, by differentiating the expression below, that the Verhulst model has a solution given by

$$N(t)=\frac{N_\infty}{\left\{1+\left[\dfrac{N_\infty}{N_0}-1\right]e^{-\gamma t}\right\}}$$

where $N=N_0$ at $t=0$.

Use your graphic calculator, or computer program to illustrate the shape of this function. For example, take $\gamma=0.03$ and

(a) $N_0=\frac{1}{2}N_\infty$

(b) $N_0=\frac{1}{4}N_\infty$

(c) $N_0=2N_\infty$

What happens as $t\to\infty$?

Returning to simple first order differential equations, you will see how to solve them in the following two examples.

Example

For $x\geq 0$, solve $\dfrac{dy}{dx}=\dfrac{x}{y}$ given that $y=1$ when $x=0$.

Solution

You can write this as

$$y\frac{dy}{dx} = x$$

and integrating both sides with respect to x.

$$\int y\frac{dy}{dx}dx = \int x\,dx$$

$$\Rightarrow \quad \int y\,dy = \int x\,dx$$

$$\Rightarrow \quad \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + K \quad (K \text{ constant})$$

Since $y = 1$ when $x = 0$,

$$\tfrac{1}{2} = 0 + K$$

$$\Rightarrow \quad K = \tfrac{1}{2}$$

and $$y^2 = x^2 + 1$$

$$\Rightarrow \quad y = \sqrt{1 + x^2}$$

Example

Solve $\frac{dy}{dx} = \frac{y}{x}$ given that $y = 1$ when $x = 1$.

Solution

As before

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\Rightarrow \quad \ln y = \ln x + C \qquad (C \text{ constant})$$

Writing $C = 1\text{n}K$,

$$\ln y = \ln x + \ln K$$

$$\Rightarrow \quad \ln y = \ln(Kx)$$

$$\Rightarrow \quad y = Kx$$

Since $y = 1$ when $x = 1$,

$$1 = K.1$$

giving $\quad K = 1$

and the solution

$$y = x\,.$$

Exercise 18E

Solve the following first order differential equations.

1. $3y\dfrac{dy}{dx} = 5x^2$ given that $y = 1$ when $x = 0$
2. $x\dfrac{dy}{dx} = 3y$ given that $y = 2$ when $x = 1$
3. $\dfrac{dy}{dx} = y^2$ given that $y = 1$ when $x = 1$
4. $\dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{1}{\left(x^2+1\right)}$ given that $y = 0$ when $x = 0$
5. $e^x\dfrac{dy}{dx} = \dfrac{x}{y}$
6. $\dfrac{dy}{dx} = \dfrac{y+1}{x+2}$
7. $x\dfrac{dy}{dx} = \dfrac{1}{y} + y$
8. During the initial stages of the growth of yeast cells in a culture, the rate of increase in the number of cells is proportional to the number of cells present.

   If $n = n(t)$ represents the number of cells at time t, show that

   $$\frac{dn}{dt} = kn$$

   for some constant k. If the cells double in unit time, determine k, and solve for n in terms of t and the initial number of cells, n_0.

18.6 Miscellaneous Exercises

1. Differentiate (a) sin $3x$ (b) sec$x$ tan$x$ (c) $\tan^2 x$.
2. Find

   (a) $\int \cos^4 x \sin x \, dx$

   (b) $\int \sin^3 x \, dx$

   (c) $\int \tan^2 x \sec^2 x \, dx$
3. The radius r of a circular ink spot, t seconds after it first appears, is given by

   $$r = \frac{1+4t}{2+t}$$

   Calculate

   (a) the time taken for the radius to double its initial value;

   (b) the rate of increase of the radius in cm s^{-1} when $t = 3$;

   (c) the value to which r tends as t tends to infinity. (AEB)

4. The volume of liquid V cm^3 in a container when the depth is x cm is given by

$$V = \frac{x^{\frac{1}{4}}}{(x+2)^{\frac{1}{2}}}, \qquad x > 0$$

(a) Find $\frac{dV}{dx}$ and determine the value of x for which $\frac{dV}{dx} = 0$.

(b) Calculate the rate of change of volume when the depth is 1 cm and increasing at a rate of 0.01 cm s^{-1}, giving your answer in cm^3 s^{-1} to three significant figures. (AEB)

5. (a) Differentiate $(1+x^3)^{\frac{1}{2}}$ with respect to x.

(b) Use the result from (a), or an appropriate substitution, to find the value of

$$\int_0^2 \frac{x}{\sqrt{(1+x^3)}}dx$$

6. Find the following integrals using the suggested substitution

(a) $\int \cos^5 x \sin x \, dx$ $\qquad (u = \cos x)$

(b) $\int \frac{x}{\sqrt{1-x^2}}dx$ $\qquad (u = 1 - x^2)$

(c) $\int \frac{e^x}{(1+e^x)^2}dx$ $\qquad (u = 1 + e^x)$

7. (a) Use integration by parts to find

$$\int x^{\frac{1}{2}} \ln x dx$$

(b) Find the solution of the differential equation

$$\frac{dy}{dx} = (xy)^{\frac{1}{2}} \ln x$$

for which $y = 1$ when $x = 1$. (AEB)

8. Find the general solution of the following first order differential equations

(a) $(x-3)\frac{dy}{dx} = y$

(b) $xy\frac{dy}{dx} = \ln x$

(c) $\frac{dx}{dy} = \frac{yx}{x-1}$