17 CARTESIAN GEOMETRY

Objectives

After studying this chapter you should

- be familiar with cartesian and parametric equations of a curve;
- be able to sketch simple curves;
- be able to recognise the rectangular hyperbola;
- be able to use the general equation of a circle;
- be able to differentiate simple functions when expressed parametrically.

17.0 Introduction

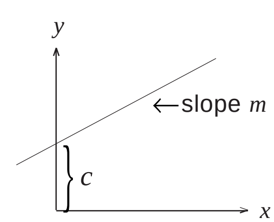
You have already met the equation of a straight line in its cartesian form - that is, y expressed as a linear function of x.

Here you will extend the analysis to other curves, including circles and hyperbolas. You will also see how to differentiate to find the gradient of a curve when it is expressed in a parametric form.

17.1 Cartesian and parametric equations of a curve

You have already met the equation of a straight line in the form

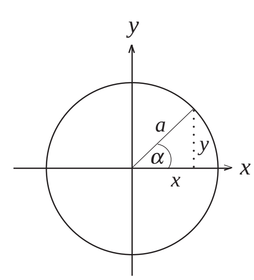
$$y = mx + c$$



Here m is the slope of the line, and c the intercept on the y-axis (see diagram opposite)

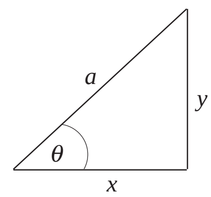
This is an example of a **cartesian equation** since it gives a relationship between the two values x and y.

Similarly, the equation of a circle, centre origin, radius a, is given by



$$x^2 + y^2 = a^2$$ (using Pythagoras)

This is again a cartesian equation, but it can also be expressed as



$$\left.\begin{aligned} x &= a\cos\theta \\ y &= a\sin\theta \end{aligned}\right\} \qquad 0 \le \theta \le 2\pi$$

This is an example of a **parametric equation** of the circle and the angle θ is the parameter.

Example

A curve is given by the parametric equation

$$\left.\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}\right\} \qquad 0 \le \theta \le 2\pi$$

Find its cartesian equation.

Solution

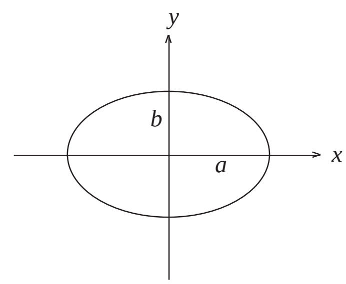
To find the cartesian equation, you need to eliminate the parameter θ; now

$$\frac{x}{a} = \cos\theta \qquad \Rightarrow \qquad \cos^2\theta = \frac{x^2}{a^2}$$

$$\frac{y}{b} = \sin\theta \qquad \Rightarrow \qquad \sin^2\theta = \frac{y^2}{b^2}$$

But $\cos^2\theta + \sin^2\theta = 1$ giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



This is in fact the equation of an ellipse as illustrated opposite when $a > b$.

*Activity 1

Use a graphic calculator or computer program to find the shape of the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

when (a) $a = 1, \quad b = 1$

(b) $a = 1, \quad b = 2$

(c) $a = 1, \quad b = 3$

(d) $a = 2, \quad b = 1$

Example

A curve is given parametrically by

$$x = t^2 \quad y = t^3$$

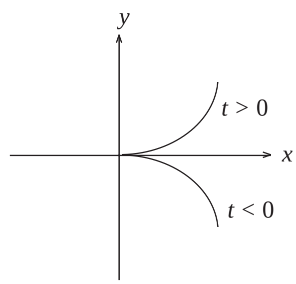
Find its cartesian equation and sketch its shape in the *xy* plane.

Solution

Eliminating the parametric *t*,

$$y = t^3 = \left(x^{\frac{1}{2}}\right)^3 = x^{\frac{3}{2}}$$

Its sketch is shown opposite; for $t > 0$ and $t < 0$. There is a cusp at the origin.



Exercise 17A

1. Find the cartesian equation of the curve when parametric equations are

   (a) $x = t^2, \quad y = 2t$

   (b) $x = 2\cos\theta, \quad y = 3\sin\theta$

   (c) $x = 2t, \quad y = \frac{1}{t}$

2. Find the stationary points of the curve when parametric equation are

   $$x = t, \quad y = t^3 - t$$

   Distinguish between them.

3. Sketch the curve given parametrically by

   $$x = t^2, \quad y = t^3$$

   Show that the equation of the normal to the curve at the point A (4, 8) is given by

   $$x + 3y - 28 = 0$$

4. A curve is given by the parametric equations; for $\theta \ge 0$

   $$x = e^{\theta} + e^{-\theta}$$
   $$y = e^{\theta} - e^{-\theta}$$

   Find its cartesian equation.

17.2 Curve sketching

You have already met many examples of curve sketching. One way is to use your graphic calculator, or a graph plotting program on a computer, but you can often determine the slope of the curve analytically. This is illustrated for the function

$$y = \frac{3(x-2)}{x(x+6)}$$

First note **special** points of the curve

(a) $y = 0 \quad \Rightarrow \quad x = 2$

(b) $y \to \pm\infty$ as $x \to 0$ and as $x \to -6$
(since $x = 0$ and -6 give zeros for the denominator)

(c) Stationary points given by $\frac{dy}{dx} = 0$ when

$$\frac{dy}{dx} = 3\left\{\frac{1.x(x+6)-(x-2)(2x+6)}{x^2(x+6)^2}\right\}$$

$$= 3\frac{\left(x^2+6x-2x^2-2x+12\right)}{x^2(x+6)^2}$$

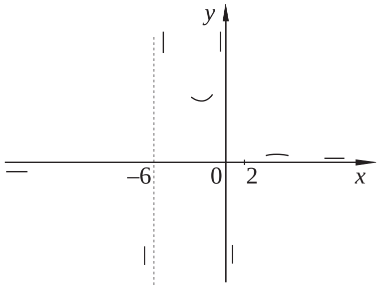
$$= 3\frac{\left(-x^2+4x+12\right)}{x^2(x+6)^2}$$

$$= -3\frac{(x+2)(x-6)}{x^2(x+6)^2}$$

This gives $x = -2$ and $x = 6$ for the stationary points.

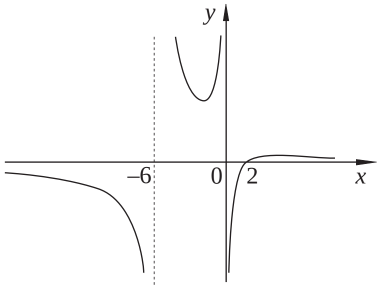
As you pass through $x = -2$, $\frac{dy}{dx}$ goes from negative to positive - hence **minimum** at $x = -2$ of value $\frac{3}{2}$.

Similarly there is a **maximum** at $x = 6$ of value $\frac{1}{6}$.



(d) As $x \to \infty$, $y \to 0$ and as $x \to -\infty$, $y \to 0$

These facts can now be plotted on a graph as shown opposite.



There is only one way that the curve can be completed. This is shown opposite.

Activity 2

Check this sketch by using a graphic calculator or graph plotting program.

Exercise 17B

In each case, without using a calculator or graph plotting program, sketch curves for the following functions. Then check your answers using a graphic calculator or graph plotting program.

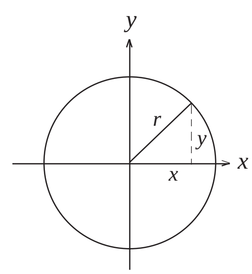
1. $y = \dfrac{2x-1}{(x-2)^2}$

2. $y = \dfrac{2}{(x+1)}$

3. $y = \dfrac{x^2+1}{\left(x^2+x+1\right)}$

4. $y = \dfrac{4x+5}{\left(x^2-1\right)}$

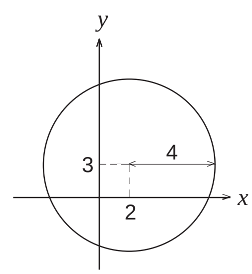
17.3 The circle



The equation of the circle, radius r, centre the origin, is clearly given by

$$x^2 + y^2 = r^2$$

How can you find the equation of a circle whose centre is not at the origin?



Suppose, you wish to find the equation of a circle, centre $x = 2, \quad y = 3$ and radius 4, as illustrated opposite.

If (x, y) is any point on the circle, then the distance between (2, 3) and (x, y) is 4 units. Hence

$$(x-2)^2 + (y-3)^2 = 4^2 = 16$$

$$\Rightarrow \quad x^2 - 4x + 4 + y^2 - 6y + 9 = 16$$

$$\Rightarrow \quad x^2 - y^2 - 4x - 6y = 3$$

Activity 3

Find the equation of a circle, centre $x = a, \quad y = b$, and radius r.

The equation in Activity 3 can be written as

$$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$$

but, given such an equation, it is not so straightforward to find the centre (a, b) and radius r. This is shown in the next example.

Example

Find the centre and radius of the circle which has the equation

$$x^2 + y^2 - 4x + 2y = 20$$

Solution

To find the centre, the L.H.S. must be written in the form

$$(x-a)^2 + (y-b)^2$$

In this case,

$$(x-2)^2 + (y+1)^2$$

since this gives

$$\left(x^2 - 4x + 4\right) + \left(y^2 + 2y + 1\right)$$

in which all the terms are correct except for the '+4' and '+1' terms. So the equation can be rewritten as

$$(x-2)^2 + (y+1)^2 - 5 = 20$$

$$\Rightarrow \quad (x-2)^2 + (y+1)^2 = 25 = 5^2$$

So it is the equation of a circle centre (2, –1), and radius 5.

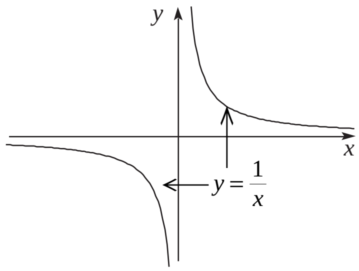
One other special curve that is of great practical importance is the **rectangular hyperbola**, which has equation

$$y = \frac{c}{x} \qquad (c \text{ constant})$$

For example if $c = 1$,

$$y = \frac{1}{x},$$

you can see that $y \to 0$ as $x \to +\infty$.



What happens to y as $x \to -\infty$?

Similarly, as $y \to \pm\infty$, $x \to 0$, and the graph is shown opposite.

Activity 4

Sketch the curve $y = f(x)$ when

(a) $f(x) = \frac{2}{x}$

(b) $f(x) = -\frac{1}{x}$

(c) $f(x) = \frac{1}{x+1}$

Exercise 17C

1. Find the equation of the circle with
   (a) centre (1, 2), radius 3
   (b) centre (0, 2), radius 2
   (c) centre (–1, –2), radius 4.

2. Find the centre and radius of the circle whose equation is
   (a) $x^2 + y^2 + 8x - 2y - 8 = 0$
   (b) $x^2 + y^2 = 16$
   (c) $x^2 + y^2 + x + 3y - 2 = 0$
   (d) $2x^2 + 2y^2 - 3x + 2y + 1 = 0$

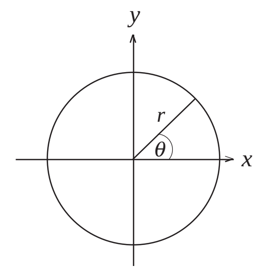
3. Find the equation of the tangent at the point (3, 1) on the circle

$$x^2 + y^2 - 4x + 10y = 8$$

*4. Find the equation of the circle which passes through the points (1, 4), (7, 5) and (1, 8).

17.4 Parametric differentiation

You have seen in section 17.1 that a parametric equation of the circle, centre origin, radius r is given by



$$x = r\cos\theta$$
$$y = r\sin\theta$$

If you wanted to find the equation of the tangent at any point $P(r\cos\theta, r\sin\theta)$, then the gradient of the tangent is given by

$$\frac{dy}{dx} = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} \quad \text{(function of function rule)}$$

$$= \frac{r\cos\theta}{-r\sin\theta}$$

$$= -\cot\theta$$

So the equation of the tangent is given by

$$(y - r\sin\theta) = -\cot\theta(x - r\cos\theta)$$

$$y\sin\theta - r\sin^2\theta = -x\cos\theta + r\cos^2\theta$$

$$y\sin\theta + x\cos\theta = r$$

Activity 5

Write the equation of the circle in the form

$$y = \sqrt{r^2 - x^2}$$

in order to find $\frac{dy}{dx}$ at the point P given by $x = x_0, \quad y = y_0$.

Hence find the equation of the tangent at P and show it is equivalent to the equation above, with $x_0 = r\cos\theta, \quad y_0 = r\sin\theta$.

Example

A curve is defined parametrically by

$$x = t^3 - 6t + 4$$
$$y = t - 3 + \frac{2}{t} \qquad (t \neq 0)$$

Find (a) the equation of the normal to the curve at the points when the curve meets the x-axis;

(b) the coordinates of their point of intersection.

Solution

Since $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$$= \left(1 - \frac{2}{t^2}\right) \Big/ \left(3t^2 - 6\right)$$

$$= \frac{\left(t^2 - 2\right)}{3t^2\left(t^2 - 2\right)}$$

$$= \frac{1}{3t^2},$$

the gradient of the normal is

$$\left(y - \left(t - 3 + \frac{2}{t}\right)\right) = -3t^2\left(x - \left(t^3 - 6t + 4\right)\right)$$

$$\frac{yt - t^2 + 3t - 2}{t} = -3t^2\left(x - t^3 + 6t - 4\right)$$

$$yt + 3t^3x = 3t^6 - 18t^4 + 12t^3 + t^2 - 3t + 2$$

The curve crosses the x-axis when $y = 0$; i.e.

$$t - 3 + \frac{2}{t} = 0$$

$\Rightarrow$ $t^2 - 3t + 2 = 0$

$\Rightarrow$ $(t - 2)(t - 1) = 0$

$\Rightarrow$ $t = 1, 2$

Equation of normal at $t = 1$ is given by

$$y + 3x = 3 - 18 + 12 + 1 - 3 + 2$$

$\Rightarrow$ $y + 3x = -3$,

and at $t = 2$,

$$2y + 24x = 192 - 288 + 96 + 4 - 6 + 2$$

$$\Rightarrow \quad y + 12x = 0.$$

These two lines intersect when

$$21x = 3 \quad \Rightarrow x = \frac{1}{3} \quad y = -4.$$

Exercise 17D

1. Show that the tangent at the point P, with parameter t, on the curve $x = 3t^2$, $y = 2t^3$ has equation

$$y = tx - t^3$$

2. The parametric equation of a curve is given by $x = \cos 2t$, $y = 4\sin t$. Sketch the curve for $0 \le t \le \frac{\pi}{2}$, and show that

$$\frac{dy}{dx} = -\operatorname{cosec} t$$

3. A curve is given by

$$x = a\cos^2 t, \; y = a\sin^3 t, \; 0 < \frac{\pi}{2}$$

when a is a positive constant. Find and simplify an expression for $\frac{dy}{dx}$ in terms of t. (AEB)

4. A curve is described parametrically by the equation

$$x = \frac{1+t}{t}, \; y = \frac{1+t^3}{t^2}$$

Find the equation of the normal to the curve at the point where $t = 2$. (AEB)

17.5 Miscellaneous Exercises

1. Sketch the curve defined parametrically by

$$x = 2 + t^2, \quad y = 4t.$$

Write down the equation of the straight line with gradient m passing through the point (1, 0). Show that this line meets the curve when

$$mt^2 - 4t + m = 0.$$

Find the values of m for which this quadratic equation has equal roots. Hence determine the equations of the tangents to the curve which pass through the point (1, 0). (AEB)

2. Determine the coordinates of the centre C and the radius of the circle with equation

$$x^2 + y^2 + 4x - 6y = 12$$

The circle cuts the x-axis at the points A and B. Calculate the area of the triangle ABC.

Calculate the area of the minor segment of the circle cut off by the chord AB, giving your answer to three significant figures.

3. Sketch the curve C defined parametrically by

$$x = t^2 - 2; \quad y = t$$

Write down the cartesian equation of the circle with centre the origin and radius r. Show that this circle meets the curve C at points whose parameter t satisfies the equation

$$t^4 - 3t^2 + 4 - r^2 = 0$$

(a) In the case $r = 2\sqrt{2}$, find the coordinates of the two points of intersection of the curve and the circle.

(b) Find the range of values of r for which the curve and the circle have exactly two points in common. (AEB)

4. A curve is defined parametrically by

$$x=\frac{2t}{1+t}, \quad y=\frac{t^2}{1+t}$$

Prove that the normal to the curve at the point $(1, \frac{1}{2})$ has equation $6y+4x=7$.

Determine the coordinates of the other point of intersection of this normal with the curve.

(AEB)

5. The parametric equations of a curve are

$$x=3(2\theta-\sin 2\theta)$$

$$y=3(1-\cos 2\theta)$$

The tangent and normal to the curve at point P

when $\theta=\frac{\pi}{4}$

meet the y-axis at L and M respectively.

Show that the area of the triangle PLM is

$$\frac{9}{4}(\pi-2)^2.$$

(AEB)