15 FURTHER TRIGONOMETRY

Objectives

After studying this chapter you should

- know all six trigonometric functions and their relationships to each other;
- be able to use trigonometric identities;
- be able to solve simple trigonometric equations;
- be able to use the sine and cosine rules.

15.0 Introduction

You will need to work both in degrees and radians, and to have a working familiarity with the sine, cosine and tangent functions, their symmetries and periodic properties. In addition, you will need freely available access to graph plotting facilities.

Three further trigonometric functions are defined as follows :

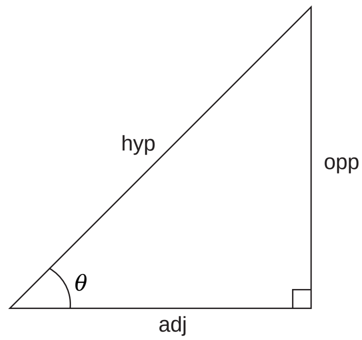
cosecant of an angle, where $\operatorname{cosec} x = \dfrac{1}{\sin x}$,

secant of an angle, where $\sec x = \dfrac{1}{\cos x}$, and

cotangent of an angle, where $\cot x = \dfrac{1}{\tan x}$.

Hence , in terms of the ratios in a right-angled triangle,

$$\operatorname{cosec}\theta = \frac{\text{hyp}}{\text{opp}},\quad \sec\theta = \frac{\text{hyp}}{\text{adj}},\quad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



and, for example,

$$\frac{1}{\sin^2 x} = \left(\frac{1}{\sin x}\right)^2 = \operatorname{cosec}^2 x, \text{ etc.}$$

Note that by Pythagoras' Theorem, $(\text{opp})^2 + (\text{adj})^2 = (\text{hyp})^2$, so dividing by $(\text{hyp})^2$ gives

$$\left(\frac{\text{opp}}{\text{hyp}}\right)^2 + \left(\frac{\text{adj}}{\text{hyp}}\right)^2 = 1^2$$

as the familiar result

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

But dividing by $(\text{adj})^2$ gives

$$\left(\frac{\text{opp}}{\text{adj}}\right)^2 + 1^2 = \left(\frac{\text{hyp}}{\text{adj}}\right)^2$$

so that $\boxed{\tan^2\theta + 1 = \sec^2\theta}$

and dividing by opp^2 gives

$$1^2 + \left(\frac{\text{adj}}{\text{opp}}\right)^2 = \left(\frac{\text{hyp}}{\text{opp}}\right)^2$$

giving $\boxed{1 + \cot^2\theta = \operatorname{cosec}^2\theta}$

These three results are sometimes refered to as the **Pythagorean identities**, and are true for **all** angles θ.

Activity 1

Use a graph-plotter to draw the graphs of $\operatorname{cosec} x$, $\sec x$ and $\cot x$ for values of x in the range $-2\pi \le x \le 2\pi$ (remember the graph plotter will work in radians). Write down in each case the period of the functions and any symmetries of the graphs.

From the definitions of cosec, sec and cot, how could you have obtained these graphs for yourself?

15.1 Identities

An identity is an equation which is true for all values of the variable. It is sometimes disinguished by the symbol $\equiv$, rather than $=$.

Example

Establish the identity $\sin A \tan A \equiv \sec A - \cos A$.

Solution

In proving results such as this sometimes it is helpful to follow this procedure: start with the left hand side (LHS), perform whatever manipulations are necessary, and work through a step at a time until the form of the right hand side (RHS) is obtained. In this case,

$$\begin{aligned}
\text{LHS} &= \sin A \tan A \\
&= \sin A \frac{\sin A}{\cos A} \\
&= \frac{\sin^2 A}{\cos A} \\
&= \frac{1-\cos^2 A}{\cos A} \qquad \text{using } \sin^2 A + \cos^2 A = 1 \\
&= \frac{1}{\cos A} - \frac{\cos^2 A}{\cos A} \\
&= \sec A - \cos A \\
&= \text{RHS}
\end{aligned}$$

Exercise 15A

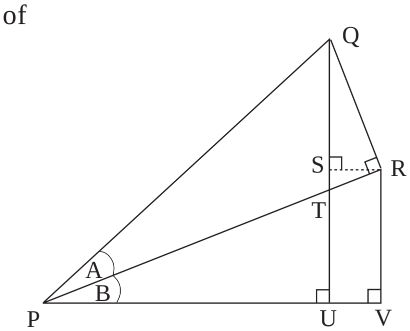
Using the basic definitions and relationships between the six trigonometric functions, prove the following identities:

1. $\sec A + \tan A = \dfrac{1+\sin A}{\cos A}$
2. $\tan A + \cot A = \sec A \operatorname{cosec} A$
3. $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$
4. $\dfrac{\operatorname{cosec}\theta - \cot\theta}{1-\cos\theta} = \operatorname{cosec}\theta$
5. $\operatorname{cosec} x - \sin x = \cos x \cot x$
6. $1 + \cos^4 x - \sin^4 x = 2\cos^2 x$
7. $\sec\theta + \tan\theta = \dfrac{\cos\theta}{1-\sin\theta}$
8. $\dfrac{\sin A \tan A}{1-\cos A} = 1 + \sec A$

15.2 The addition formulae

A proof of the formula for sin ($A+B$) will be given here.

Consider the diagram opposite which illustrates the geometry of the situation.



In triangle PQU,

$$\sin(A+B) = \frac{QU}{PQ}$$

$$= \frac{QS+SU}{PQ}$$

$$= \frac{QS}{PQ} + \frac{RV}{PQ} \quad (\text{since } SU = RV)$$

Now notice that, since $P\hat{T}U = 90 - B$, $S\hat{T}R = 90 - B$ also,

and so $S\hat{Q}R = B$ (since $T\hat{R}Q = 90°$).

Then in triangle QRS

$$QS = QR \cos B.$$

Also, in triangle PVR,

$$RV = PR \sin B.$$

You now have

$$\sin(A+B) = \frac{QR}{PQ}\cos B + \frac{PR}{PQ}\sin B.$$

But in triangle PQR, $\frac{QR}{PQ} = \sin A$ and $\frac{PR}{PQ} = \cos A$ so,

$$\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B}$$

Rather than reproducing similar proofs for three more formulae, the following approach assumes this formula for sin$(A+B)$ and uses prior knowledge of the sine and cosine functions.

Given $\sin(A+B) = \sin A\cos B + \cos A\sin B$, replacing B by $-B$ throughout gives

$$\sin(A-B) = \sin A\cos(-B) + \cos A\sin(-B) .$$

Now $\cos(-B) = +\cos B$ and $\sin(-B) = -\sin B$, so

$$\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B}$$

Next use the fact that $\cos\theta = \sin(90-\theta)$, so that

$$\begin{aligned}\cos(A+B) &= \sin(90-(A+B)) \\ &= \sin((90-A)-B) \\ &= \sin(90-A)\cos B - \cos(90-A)\sin B\end{aligned}$$

from the second result just obtained. And since

$$\cos A = \sin(90-A),\ \sin A = \cos(90-A),$$

it follows that

$$\boxed{\cos(A+B) = \cos A\ \cos B - \sin A\ \sin B}$$

Replacing B by $-B$, as before, this gives the fourth result

$$\boxed{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

Example

Show that $\sin 15° = \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$.

Solution

$$\begin{aligned}\sin 15° &= \sin(60°-45°) \\ &= \sin 60° \cos 45° - \cos 60° \sin 45°\end{aligned}$$

Now $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$ and $\cos 45° = \frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$

giving

$$\sin 15° = \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} - \frac{1}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}\left(\sqrt{3}-1\right)$$

Exercise 15B

1. Using the values $\cos 0° = 1$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\cos 45° = \frac{1}{\sqrt{2}}$, $\cos 60° = \frac{1}{2}$ and $\cos 90° = 0$ and related values of sine and tangent,determine, in similar form,

   (a) $\sin 75°$ (b) $\cos 15°$ (c) $\cos 105°$
   (d) $\cos 75°$ (e) $\tan 75°$ (f) $\tan 105°$
   (g) $\sin 255°$ (h) $\cos 285°$ (i) $\cot 75°$

2. Show that
   $$\cos(45°-A) - \cos(45°+A) = \sqrt{2}\sin A$$

3. Show that
   $$\sin x + \sin\left(x + \frac{2}{3}\pi\right) + \sin\left(x + \tfrac{4}{3}\pi\right) = 0$$

4. Given that $\sin(A+B) = 3\sin(A-B)$, show that $\tan A = 2\tan B$

5. Show that
   $$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y.$$

15.3 Further identities

Now

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$
$$= \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$
$$= \frac{\left(\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}\right)}{\left(\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}\right)}$$

Here, you can divide every term of the fraction by cos *A* cos *B*, giving

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\ \tan B}$$

Replacing *B* by –*B* gives

$$\tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\ \tan B}$$

Multiple angles

If, in the formula for sin (*A*+*B*), you put *B* = *A*, then you get

$$\sin(A+A) = \sin A\cos A+\cos A\sin A$$

or

$$\sin 2A = 2\sin A\cos A$$

Activity 2

Use the addition formulae to find expressions for

(a) cos 2*A* in terms of cos *A* and sin *A;*
(b) cos 2*A* in terms of cos *A* only;
(c) cos 2*A* in terms of sin *A* only;
(d) tan 2*A* in terms of tan *A* only;
(e) sin 3*A* in terms of powers of sin *A* only;
(f) cos 3*A* in terms of powers of cos *A* only.

Example

Establish the identity $\dfrac{\cos 2A + \sin 2A - 1}{\cos 2A - \sin 2A + 1} = \tan A$.

Solution

Here, you should first look to simplify the numerator and denominator by using the identities for cos 2*A* and sin 2*A*.

In the numerator, re-writing $\cos 2A$ as $1 - 2\sin^2 A$ will help cancel the –1 on the end; while $\cos 2A = 2\cos^2 A - 1$ will be useful in the denominator.

$$\begin{aligned}
\text{LHS} &= \frac{1 - 2\sin^2 A + 2\sin A\cos A - 1}{2\cos^2 A - 1 - 2\sin A\cos A + 1} \\
&= \frac{2\sin A\cos A - 2\sin^2 A}{2\cos^2 A - 2\sin A\cos A} \\
&= \frac{2\sin A(\cos A - \sin A)}{2\cos A(\cos A - \sin A)} \\
&= \frac{\sin A}{\cos A} \\
&= \tan A \\
&= \text{RHS}
\end{aligned}$$

Example

Show that $\sec^2\theta + \operatorname{cosec}^2\theta = 4\operatorname{cosec}^2 2\theta$.

Solution

To begin with, for shorthand write $s = \sin\theta$ and $c = \cos\theta$.

Then

$$\begin{aligned}
\text{LHS} &= \frac{1}{c^2} + \frac{1}{s^2} \\
&= \frac{s^2 + c^2}{s^2c^2} \\
&= \frac{1}{s^2c^2}
\end{aligned}$$

Now notice that $2sc = \sin 2\theta$,

so $4s^2c^2 = \sin^2 2\theta$ and $s^2c^2 = \tfrac{1}{4}\sin^2 2\theta$, giving

$$\text{LHS} = \frac{1}{\tfrac{1}{4}\sin^2 2\theta} = 4\operatorname{cosec}^2 2\theta = \text{RHS}.$$

Example

Prove the identity $\dfrac{\cos A + \sin A}{\cos A - \sin A} = \sec 2A + \tan 2A$

Solution

This problem is less straightforward and requires some ingenuity. It helps to note that the

$$\text{RHS} = \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$$

with a common denominator of $\cos 2A$. One formula for $\cos 2A$ is

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= (\cos A - \sin A)(\cos A + \sin A)$$

by the difference of two squares.

Hence

$$\text{LHS} = \frac{\cos A + \sin A}{\cos A - \sin A}$$

$$= \frac{(\cos A + \sin A)}{(\cos A - \sin A)} \times \frac{(\cos A + \sin A)}{(\cos A + \sin A)}$$

This is done to get the required form in the denominator.

$$\text{LHS} = \frac{\cos^2 A + 2\sin A\cos A + \sin^2 A}{\cos^2 A - \sin^2 A.}$$

$$= \frac{\left(\cos^2 A + \sin^2 A\right) + 2\sin A\cos A}{\cos 2A}$$

$$= \frac{1 + \sin 2A}{\cos 2A}$$

$$= \frac{1}{\cos 2A} + \frac{\sin 2A}{\cos 2A}$$

$$= \sec 2A + \tan 2A$$

$$= \text{RHS.}$$

Exercise 15C

Prove the following identities in Questions 1 to 6.

1. $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$
2. $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$
3. $\dfrac{\sin 2A + \cos 2A + 1}{\sin 2A - \cos 2A + 1} = \cot A$
4. $\cot x - \operatorname{cosec} 2x = \cot 2x$
5. $\dfrac{\sin 3A + \sin A}{2\sin 2A} = \cos A$
6. $\dfrac{\cos 3\theta - \sin 3\theta}{1 - 2\sin 2\theta} = \cos\theta + \sin\theta$

   [You may find it useful to refer back to the results of Activity 2 for Questions 5 and 6.]
7. Use the fact that $4A = 2\times 2A$ to show that

   $\dfrac{\sin 4A}{\sin A} = 8\cos^3 A - 4\cos A.$
8. By writing $t = \tan\theta$ show that

   $\tan(\theta + 45°) + \tan(\theta - 45°) = \dfrac{1+t}{1-t} - \dfrac{1-t}{1+t}.$

   Hence show that

   $\tan(\theta + 45°) + \tan(\theta - 45°) = 2\tan 2\theta.$
9. Using $t = \tan\theta$, write down $\tan 2\theta$ in terms of t. Hence prove the identities

   (a) $\cot\theta - \tan\theta = 2\cot 2\theta$

   (b) $\cot 2\theta + \tan\theta = \operatorname{cosec} 2\theta$
10. Write down $\cos 4x$ in terms of $\cos 2x$, and hence in terms of $\cos x$ show that

    $\cos 4x + 4\cos 2x = 8\cos^4 x - 3$
11. Prove the identity

    $\dfrac{\sin 4A + \cos A}{\cos 4A + \sin A} = \sec 3A + \tan 3A$

*15.4 Sum and product formulae

You may recall that

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

Adding these two equations gives

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B \qquad (1)$$

Call $C = A + B$ and $D = A - B$,

then $C + D = 2A$ and $C - D = 2B$. Hence

$$A = \frac{C+D}{2},\ B = \frac{C-D}{2}$$

and (1) can be written as

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

This is more easily remembered as

'sine plus sine = twice sine(half the sum) cos(half the difference)'

Activity 3

In a similar way to above, derive the formulae for

(a) $\sin C - \sin D$ (b) $\cos C + \cos D$ (c) $\cos C - \cos D$

By reversing these formulae, write down further formulae for

(a) $2\sin E\cos F$ (b) $2\cos E\cos F$ (c) $2\sin E\sin F$

Example

Show that $\cos 59° + \sin 59° = \sqrt{2}\cos 14°$.

Solution

Firstly, $\sin 59° = \cos 31°$, since $\sin\theta = \cos(90 - \theta)$

So
$$\begin{aligned}\text{LHS} &= \cos 59° + \cos 31° \\ &= 2\cos\left(\frac{59+31}{2}\right)\cos\left(\frac{59-31}{2}\right) \\ &= 2\cos 45° \times \cos 14° \\ &= 2 \times \frac{\sqrt{2}}{2}\cos 14° \\ &= \sqrt{2}\cos 14° \\ &= \text{RHS}\end{aligned}$$

Example

Prove that $\sin x + \sin 2x + \sin 3x = \sin 2x(1 + 2\cos x)$.

Solution

$$\begin{aligned}\text{LHS} &= \sin 2x + (\sin x + \sin 3x) \\ &= \sin 2x + 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right) \\ &= \sin 2x + 2\sin 2x\cos(-x) \\ &= \sin 2x(1 + 2\cos x) \quad \text{since } \cos(-x) = \cos x.\end{aligned}$$

Example

Write $\cos 4x \cos x - \sin 6x \sin 3x$ as a product of terms.

Solution

Now $\cos 4x \cos x = \frac{1}{2}\{\cos(4x+x)+\cos(4x-x)\}$

$$= \frac{1}{2}\cos 5x + \frac{1}{2}\cos 3x$$

and $\sin 6x \sin 3x = \frac{1}{2}\{\cos(6x-3x)-\cos(6x+3x)\}$

$$= \frac{1}{2}\cos 3x - \frac{1}{2}\cos 9x.$$

Thus, LHS $= \frac{1}{2}\cos 5x + \frac{1}{2}\cos 3x - \frac{1}{2}\cos 3x + \frac{1}{2}\cos 9x$

$$= \frac{1}{2}(\cos 5x + \cos 9x)$$

$$= \frac{1}{2} \times 2\cos\left(\frac{5x+9x}{2}\right)\cos\left(\frac{5x-9x}{2}\right)$$

$$= \cos 7x \cos 2x.$$

The sum formulae are given by

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

and the product formulae by

$$\sin A \cos B = \tfrac{1}{2}(\sin(A+B)+\sin(A-B))$$

$$\cos A \cos B = \tfrac{1}{2}(\cos(A+B)+\cos(A-B))$$

$$\sin A \sin B = \tfrac{1}{2}(\cos(A-B)-\cos(A+B))$$

*Exercise 15D

1. Write the following expressions as products:

   (a) $\cos 5x - \cos 3x$ (b) $\sin 11x - \sin 7x$

   (c) $\cos 2x + \cos 9x$ (d) $\sin 3x + \sin 13x$

   (e) $\cos\frac{2\pi}{15} + \cos\frac{14\pi}{15} + \cos\frac{4\pi}{15} + \cos\frac{8\pi}{15}$

   (f) $\sin 40° + \sin 50° + \sin 60°$

   (g) $\cos 114° + \sin 24°$

2. Evaluate in rational/surd form

   $\sin 75° + \sin 15°$

3. Write the following expressions as sums or differences:

   (a) $2\cos 7x \cos 5x$

   (b) $2\cos\left(\frac{1}{2}x\right)\cos\left(\frac{5x}{2}\right)$

   (c) $2\sin\left(\frac{\pi}{4} - 3\theta\right)\cos\left(\frac{\pi}{4} + \theta\right)$

   (d) $2\sin 165° \cos 105°$

4. Establish the following identities:

   (a) $\cos\theta - \cos 3\theta = 4\sin^2\theta\cos\theta$

   (b) $\sin 6x + \sin 4x - \sin 2x = 4\cos 3x \sin 2x \cos x$

   (c) $\dfrac{2\sin 4A + \sin 6A + \sin 2A}{2\sin 4A - \sin 6A - \sin 2A} = \cot^2 A$

   (d) $\dfrac{\sin(A+B) + \sin(A-B)}{\cos(A+B) - \cos(A-B)} = -\cot B$

   (e) $\dfrac{\cos(\theta+30°) + \cos(\theta+60°)}{\sin(\theta+30°) + \sin(\theta+60°)} = \dfrac{1-\tan\theta}{1+\tan\theta}$

5. Write $\cos 12x + \cos 6x + \cos 4x + \cos 2x$ as a product of terms.

6. Express $\cos 3x \cos x - \cos 7x \cos 5x$ as a product of terms.

15.5 General formula

For this next activity you will find it very useful to have a graph plotting facility. Remember, you will be working in radians.

Activity 4

Sketch the graph of a function of the form

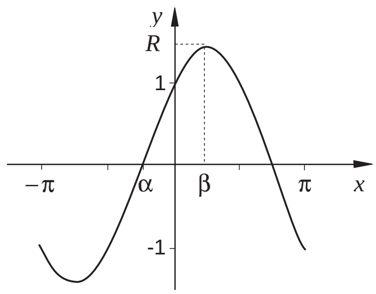
$$y = a\sin x + b\cos x$$

(where a and b are constants) in the range $-\pi \le x \le \pi$.
From the graph, you must identify the amplitude of the function and the x-coordinates of

(i) the crossing-point on the x-axis nearest to the origin, and

(ii) the first maximum of the function

as accurately as you can.



An example has been done for you; for $y = \sin x + \cos x$, you can see that amplitude ≈ 1.4

crossing-point nearest to O at $x = \alpha = -\frac{\pi}{4}$

maximum occurs at $x = \beta = \frac{\pi}{4}$

Try these for yourself :

(a) $y = 3\sin x + 4\cos x$ (b) $y = 12\cos x - 5\sin x$

(c) $y = 9\cos x + 12\sin x$ (d) $y = 15\sin x - 8\cos x$

(e) $y = 2\sin x + 5\cos x$ (f) $y = 3\cos x - 2\sin x$

In each case, make a note of

R, the amplitude;

α, the crossing-point nearest to O;

β, the x-coordinate of the maximum.

In each example above, you should have noticed that the curve is itself a sine/cosine 'wave', each of which can be obtained from the curves of either $y = \sin x$ or $y = \cos x$ by means of two simple transformations (taken in any order):

1. a **stretch** parallel to the *y*-axis by a factor of *R*, the amplitude, and

2. a **translation** parallel to the *x*-axis by either α or β (depending on whether you wish to start with sin *x* or cos *x* as the original function).

Consider, for example $y = \sin x + \cos x$. This can be written in the form $y = R\sin(x + \alpha)$, since

$$R\sin(x+\alpha) = R\{\sin x\cos\alpha + \cos x\sin\alpha\}$$
$$= R\cos\alpha\sin x + R\sin\alpha\cos x$$

This expression should be the same as $\sin x + \cos x$.

Thus

$R\cos\alpha = 1$ and $R\sin\alpha = 1$

Dividing these terms gives

$$\tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$$

Squaring and adding the two terms gives

$$R^2\cos^2\alpha + R^2\sin^2\alpha = 1^2 + 1^2$$

$$R^2\left(\cos^2\alpha + \sin^2\alpha\right) = 2$$

Since $\cos^2\alpha + \sin^2\alpha = 1$,

$$R^2 = 2 \Rightarrow R = \sqrt{2} \quad \text{(negative root does not make sense)}$$

Thus

$$\sin x + \cos x = \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right).$$

Activity 5

Express the function $\sin x + \cos x$ in the form

$$\sin x + \cos x = R\cos(x - \alpha).$$

Find suitable values for R and α using the method shown above.

Another way of obtaining the result in Activity 5 is to note that

$$\sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right)$$

so that

$$\begin{aligned}\sin x + \cos x &= \sqrt{2}\sin\left(x + \tfrac{\pi}{4}\right)\\ &= \sqrt{2}\cos\left(\tfrac{\pi}{2} - \left(x + \tfrac{\pi}{4}\right)\right)\\ &= \sqrt{2}\cos\left(\tfrac{\pi}{4} - x\right)\\ &= \sqrt{2}\cos\left(x - \tfrac{\pi}{4}\right)\end{aligned}$$

since $\cos(-\theta) = \cos\theta$.

Example

Write $7\sin x - 4\cos x$ in the form $R\sin(x-\alpha)$

where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Solution

Assuming the result,

$$7\sin x - 4\cos x = R\sin(x-\alpha)$$
$$= R\sin x\cos\alpha - R\cos x\sin\alpha$$

To satisfy the equation, you need

$$R\cos\alpha = 7$$
$$R\sin\alpha = 4$$

Squaring and adding, as before, gives

$$R = \sqrt{7^2 + 4^2} = \sqrt{65}.$$

Thus

$$\cos\alpha = \frac{7}{\sqrt{65}},\ \sin\alpha = \frac{4}{\sqrt{65}}\ \left(\text{or } \tan\alpha = \frac{4}{7}\right)$$
$$\Rightarrow \alpha = 0.519 \text{ radians,}$$

so $$7\sin x - 4\cos x = \sqrt{65}\sin(x - 0.519)$$

Exercise 15E

Write (in each case, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$)

1. $3\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$
2. $4\cos x + 3\sin x$ in the form $R\cos(x-\alpha)$
3. $15\sin x - 8\cos x$ in the form $R\sin(x-\alpha)$
4. $6\cos x - 2\sin x$ in the form $R\cos(x+\alpha)$
5. $20\sin x - 21\cos x$ in the form $R\sin(x-\alpha)$
6. $14\cos x + \sin x$ in the form $R\cos(x-\alpha)$
7. $2\cos 2x - \sin 2x$ in the form $R\cos(2x+\alpha)$
8. $3\cos\frac{1}{2}x + 5\sin\frac{1}{2}x$ in the form $R\sin\left(\frac{1}{2}x+\alpha\right)$

15.6 Equations in one function

In Chapter 10 you looked at equations of the form

$$a\sin(bx+c)=d$$

for constants *a, b, c, d,* or similar equations involving cos or tan.

In this and the following sections you will be introduced to a variety of different types of trigonometric equation and the appropriate ways of solving them within a given range.

Here, you will be asked only to solve polynomials in one function. It is important, therefore, that you are able to determine factors of polynomials and use the quadratic formula when necessary.

Example

Solve $2\sin^2\theta+3\sin\theta=2$ for values of θ between $0°$ and $360°$.

Solution

Rearrange the equation as $2\sin^2\theta+3\sin\theta-2=0$, which is a quadratic in $\sin\theta$. This factorises as

$$(2\sin\theta-1)(\sin\theta+2)-0$$

giving

(a) $2\sin\theta-1=0 \Rightarrow \sin\theta=\frac{1}{2} \Rightarrow \theta=30°, 150°$

or

(b) $\sin\theta+2=0 \Rightarrow \sin\theta=-2$ which has no solutions.

Exercise 15F

Solve the following equations for values of *x* in the range given:

1. $4\sin^2 x-\sin x-3=0,\ 0\le x\le 2\pi$
2. $6\cos^2 x+\cos x=1,\ -180°\le x\le 180°$
3. $\tan^2 x+3\tan x-10=0,\ 0°\le x\le 360°$
4. $\cos^2 x=2\cos x+1,\ 0°\le x\le 180°$
5. $\tan^4 x-4\tan^2 x+3=0,\ 0\le x\le \pi$
6. $\frac{1}{2}\sec^2 x=\sec x+2,\ 0\le x\le \pi$
7. Use the factor theorem to factorise

   $$6c^3-19c^2+c+6=0$$

   and hence solve

   $$6\cos^3 x-19\cos^2 x+\cos x+6=0$$

   for $0\le x\le \pi$

15.7 Equations in two functions reducible to one

Equations involving two (or more) trigonometric functions cannot, in general, be solved by the simple methods you have encountered up to now. However, many such equations can be tackled using some of the basic identities introduced in the first part of this chapter.

Example

Solve $5\sin\theta = 2\cos\theta$ for $0 \le \theta \le 2\pi$.

Solution

Dividing both sides by $\cos\theta$ assuming $\cos\theta \ne 0$ gives

$$5\tan\theta = 2$$
$$\Rightarrow \quad \tan\theta = 0.4$$
$$\Rightarrow \quad \theta = 0.381,\ 3.52$$

Example

Solve $2\sec^2 x + 3\tan x - 4 = 0$ for $0° \le x \le 180°$.

Solution

From earlier work, $\sec^2 x = 1 + \tan^2 x$, leading to

$$2 + 2\tan^2 x + 3\tan x - 4 = 0$$
$$\Rightarrow \quad 2\tan^2 x + 3\tan x - 2 = 0$$
$$\Rightarrow \quad (2\tan x - 1)(\tan x + 2) = 0$$

giving

(a) $2\tan x - 1 = 0 \Rightarrow \tan x = \frac{1}{2} \Rightarrow x = 26.6°$

or

(b) $\tan x + 2 = 0 \Rightarrow \tan x = -2 \Rightarrow x = 116.6°$

Example

Solve $3\sin 2\theta = 5\cos\theta$ for $0° \le \theta \le 180°$

Solution

Since $\sin 2\theta = 2\sin\theta\cos\theta$, the equation reduces to

$$6\sin\theta\cos\theta = 5\cos\theta.$$

Method 1 – divide by $\cos\theta$ to get

$$\sin\theta = \frac{5}{6}$$

$$\Rightarrow \theta = 56.4°,\ 123.6°$$

Method 2 – factorise to give

$$6\sin\theta\cos\theta - 5\cos\theta = 0$$

$$\cos\theta(6\sin\theta - 5) = 0$$

giving

(a) $\cos\theta = 0 \Rightarrow \theta = 90°$

or

(b) $\sin\theta = \frac{5}{6} \Rightarrow \theta = 56.4°, 123.6°$

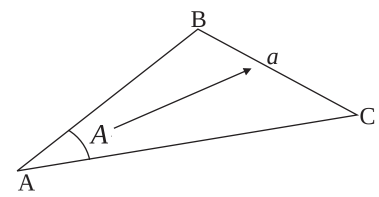
You should see the error in **Method 1**, which throws away the solution for $\cos\theta = 0$. Division can only be done provided that the quantity concerned is **not** zero. [You might like to check back in the first example to see that exactly the same division was quite legitimate in that situation.]

Exercise 15G

Solve the following equations in the required domain :

1. $2\sin^2\theta + 5\cos\theta + 1 = 0$ $\quad -\pi \le \theta \le \pi$
2. $2\sin 2\theta = \tan\theta$ $\quad 0° \le \theta \le 180°$
3. $2\operatorname{cosec} x = 5\cot x$ $\quad 0° \le x \le 180°$
4. $3\cos\theta = 2\cos 2\theta$ $\quad 0° \le \theta \le 360°$
5. $\sin x + \frac{1}{2}\sin 2x = 0$ $\quad 0 \le x \le 2\pi$
6. $6\cos\theta - 1 = \sec\theta$ $\quad 0° \le \theta \le 180°$
7. $\tan^2 x + 3\sec x = 0$ $\quad 0 \le x \le 2\pi$
8. $6\tan^2 A = 4\sin^2 A + 1$ $\quad 0° \le A \le 360°$
9. $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$ $\quad 0 \le \theta \le 2\pi$
10. $\operatorname{cosec} x = \sqrt{3}\sec^2 x$ $\quad 0 \le x \le \pi$
11. $\sec^4\theta + 2 = 6\tan^2\theta$ $\quad 0° \le \theta \le 180°$
12. $\cos 2\theta\cos\theta = \sin 2\theta\sin\theta$ $\quad -180° \le \theta \le 180°$

*15.8 The sine rule



In this section and the next you will be introduced to the sine and cosine rules for use in any triangle. Before you start, you should be aware of the convention for referring to the sides and angles of a triangle.

In triangle ABC shown opposite, the angles are labelled as the vertices at which they occur, and are denoted by capital letters, so that

angle ABC = B.

Lower case letters refer to the sides of the triangle, so that

side BC = a,

with the convention that a is opposite angle A (as shown), b opposite angle B, and so on.

Activity 6 Finding a rule for sides and angles

For this activity you will need a ruler measuring in mm, an angle measurer or protractor, and a calculator.

Draw four different shaped triangles. (You should include some obtuse-angled triangles).

Label the vertices A, B and C and the opposite sides a, b and c corresponding to the angles.

Measure the size of angles A, B and C (an accuracy to the nearest half-degree should be possible) and the lengths of the sides a, b and c to the nearest mm. Then for each triangle, evaluate

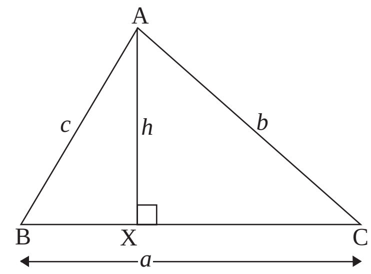
$$\frac{a}{\sin A}, \frac{b}{\sin B} \text{ and } \frac{c}{\sin C}.$$

What do you notice?

A proof of the sine rule

Oddly enough, in order to work with the sine and cosine functions in a non right-angled triangle it is necessary to create a right-angle.

In the triangle ABC, a perpendicular has been drawn from A to BC, meeting BC at the point X at 90°.



Here, then, AX is the height of ABC and BC is the base.

In triangle ABX, $h = c\sin B$.

In triangle AXC, $h = b\sin C$.

By putting the two formulas for h together

$$c\sin B = b\sin C \quad \text{or} \quad \frac{c}{\sin C} = \frac{b}{\sin B}$$

If side AC had been taken as the base, the relationship

$\frac{c}{\sin C} = \frac{a}{\sin A}$ would have been obtained, and taking AB as base

would have given $\frac{a}{\sin A} = \frac{b}{\sin B}$.

Together, the set of equations obtained is

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

This is called the **sine rule**, relating the sides of any triangle to the sines of its angles.

(Remember: this rule applies to **any** triangle, with or without a right angle).

Example

In triangle ABC, $A = 40°$ and $a = 17$ mm; $c = 11$ mm. Find b, B and C.

Solution

In attempting to solve a problem of this sort a sketch is necessary.

In the equation $\frac{a}{\sin A} = \frac{c}{\sin C}$, three of the four quantities are known, or can be found. The fourth, $\sin C$, can be calculated, and hence C.

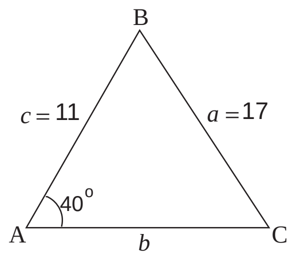
Substituting,

$$\frac{17}{\sin 40^\circ} = \frac{11}{\sin C}.$$

Rearranging,

$$\sin C = \frac{11\sin 40^\circ}{17} = 0.415921\ldots$$

$$\Rightarrow \quad C = 24.6^\circ.$$



Knowing A and C,

$$B = 180^\circ - A - C$$
$$= 115.4^\circ.$$

b can now be found using

$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad \left(\text{or } \frac{c}{\sin C}\right).$$

Substituting,

$$\frac{b}{\sin 115.4^\circ} = \frac{17}{\sin 40^\circ}.$$

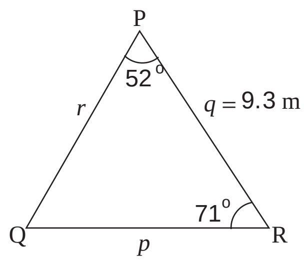
Rearranging,

$$b = \frac{17\sin 115.4^\circ}{\sin 40^\circ}$$
$$= 23.9 \text{ mm} \quad \text{(to 3 significant figures).}$$

Example

In triangle PQR, $P = 52°$, $R = 71°$ and $q = 9.3$ m.

Find Q, p and r.



Solution

Firstly, $Q = 180° - 52° - 71° = 57°$.

Next, using $\dfrac{q}{\sin Q} = \dfrac{p}{\sin P}$,

$$\frac{9.3}{\sin 57^\circ} = \frac{p}{\sin 52^\circ}.$$

Rearranging,

$$p = \frac{9.3\sin 52^\circ}{\sin 57^\circ} = 8.74 \text{ m}.$$

Also, $\dfrac{q}{\sin Q} = \dfrac{r}{\sin R}$,

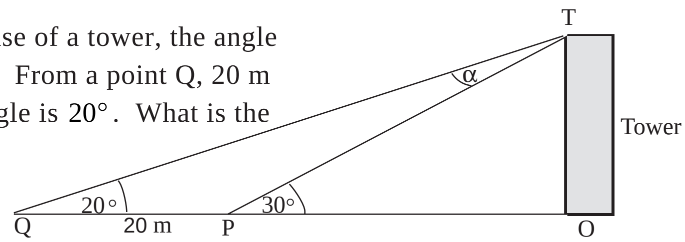
giving $\dfrac{9.3}{\sin 57^\circ} = \dfrac{r}{\sin 71^\circ}$.

Rearranging,

$$r = \frac{9.3 \sin 71^\circ}{\sin 57^\circ} = 10.5 \text{ m} \quad \text{(to 3 significant figures).}$$

Example

From a point P on the same level as the base of a tower, the angle of elevation of the top of the tower is 30°. From a point Q, 20 m further away than P from the tower the angle is 20°. What is the height of the tower?



Solution

TP is first found by using the sine rule in triangle QPT;

$$\frac{\text{TP}}{\sin 20^\circ} = \frac{\text{QP}}{\sin \alpha}$$

But $20^\circ + \alpha = 30^\circ \Rightarrow \alpha = 10^\circ$

$$\text{TP} = \frac{20 \sin 20^\circ}{\sin 10^\circ} = 39.39 \text{ m}.$$

Finally, from triangle TOP,

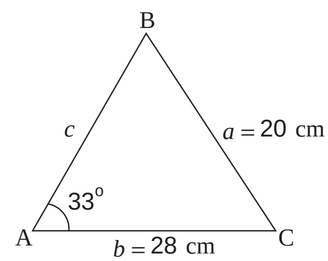
height, $\text{TO} = \text{TP} \sin 30^\circ = 39.39 \times \sin 30^\circ = 19.7 \text{ m}$ (to 3 significant figures).

A possible difficulty

Example

Solve the triangle ABC, given $A = 33^\circ$, $a = 20$ cm and $b = 28$ cm.

In this context 'solve' means 'find all the other sides and angles not already given'.



Solution

Now $\dfrac{a}{\sin A} = \dfrac{b}{\sin B}$ gives $\dfrac{20}{\sin 33^\circ} = \dfrac{28}{\sin B}$.

Hence $\sin B = \dfrac{28 \sin 33°}{20} = 0.762\ 495$,

which gives $B = 49.7°$. But B could be obtuse, and another possible solution is given by

$$B = 180° - 49.7° = 130.3°.$$

Now if $B = 49.7°$,

$$C = 180° - 33° - 49.7° = 97.3°,$$

and $\dfrac{20}{\sin 33°} = \dfrac{c}{\sin 97.3°}$

gives $c = \dfrac{20 \sin 97.3°}{\sin 33°} = 36.4 \text{ cm } (3 \text{ s.f.})$.

But if $B = 130.3°$,

$$C = 180° - 33° - 130.3° = 16.7°$$

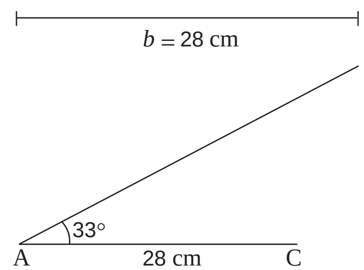
and $c = \dfrac{20 \sin 16.7°}{\sin 33°} = 10.6 \text{ cm } (3\text{s.f.})$.

So there appear to be two possible solutions.

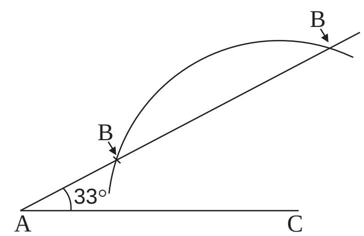
Does this make sense?

In order to visualise the reason for this ambiguity, imagine trying to draw the triangle as described:

$$A = 33°,\ a = 20,\ b = 28.$$



1. Draw the longest side first : $b = 28$.
2. Measure an angle of 33° at A - the position of B on this line is not yet known.
3. $CB = 20$, so B is 20 cm from C and somewhere on the line from A. Now all possible positions of a point B such that $BC = 20$ lie on a circle, centre at C, and radius 20. Part of this circle is drawn on the diagram.



You will see that the circle and line intersect in two points corresponding to the **two** possible positions of B.

This situation arises when you are given two sides and a non-included angle (i.e. not the angle between them) because the triangle is not necessarily uniquely defined by the information given. It is called the **ambiguous case** , and you must watch out for it when using the sine rule to find angles.

Turn back to the first example in this chapter and see if you can decide why the problem did not arise there.

*Exercise 15H

In the following triangles, find the sides and angles not given. Give your answers to 1 d.p. for angles and 3 s.f. for sides where appropriate.

1. In triangle LMN, $m = 32$m, $M = 16°$ and $N = 56.7°$.
2. In triangle XYZ, $X = 120°$, $x = 11$ cm and $z = 5$ cm.
3. In triangle ABC, $A = 49°$, $a = 127$ m, and $c = 100$ m.
4. In triangle PQR, $R = 27°$, $p = 9.2$ cm and $r = 8.3$ cm.
5. In triangle DEF, $E = 81°$, $F = 62°$ and $d = 4$ m.
6. In triangle UVW, $u = 4.2$ m, $w = 4$ m and $W = 43.6°$.

*15.9 The cosine rule

Activity 7

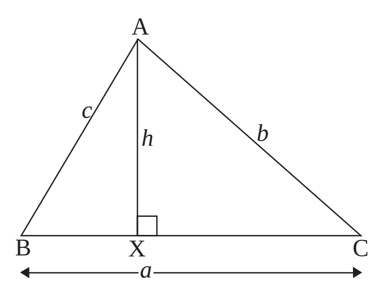
Why is it that the sine rule does not enable you to solve triangles ABC and XYZ when

(a) in triangle ABC you are given :

$A = 35°$, $b = 84$ cm and $c = 67$ cm;

(b) in triangle XYZ you are given :

$x = 43$ m, $y = 60$ m and $z = 81$ m?



As with the introduction of the sine rule, it is necessary to create a right-angle in order to establish the cosine rule. Again, it is not important which side is taken as base.

The activity above shows the need for another rule in order to 'solve' triangles.

Two applications of Pythagoras' Theorem give

$$c^2 = h^2 + \text{BX}^2 \text{ in triangle ABX,}$$

and $$b^2 = h^2 + \text{XC}^2 \text{ in triangle ABC.}$$

Rearranging in terms of h^2,

$$c^2 - \text{BX}^2 = b^2 - \text{XC}^2$$

i.e. $$c^2 = b^2 + \text{BX}^2 - \text{XC}^2.$$

Now $\mathrm{BX}^2 - \mathrm{XC}^2$ is the difference of two squares and can be factorised as $(\mathrm{BX}+\mathrm{XC})(\mathrm{BX}-\mathrm{XC})$.

Notice that $\mathrm{BX}+\mathrm{XC} = a$, and

$$\mathrm{BX}-\mathrm{XC} = \mathrm{BX}+\mathrm{XC}-2\mathrm{XC}$$
$$= a - 2\mathrm{XC}.$$

Whereas, in triangle AXC, $\mathrm{XC} = b\cos C$.

Putting all these together gives

$$c^2 = b^2 + (\mathrm{BX}+\mathrm{XC})\,(\mathrm{BX}-\mathrm{XC})$$

$$\Rightarrow \quad c^2 = b^2 + a(a - 2b\cos C)$$

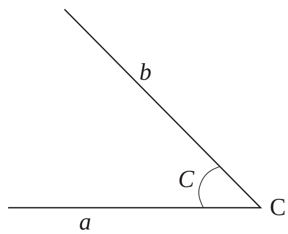
$$\Rightarrow \quad \boxed{c^2 = a^2 + b^2 - 2ab\cos C.}$$

Thus, given two sides of a triangle and the included angle, the **cosine rule** enables you to find the remaining side. Similarly

$$a^2 = b^2 + c^2 - 2bc\cos A$$

and $$b^2 = c^2 + a^2 - 2ca\cos B$$

are equivalent forms of the **cosine rule**, which you could have found by choosing one of the other sides as base in the diagram.



Formula for the cosine of an angle

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\Rightarrow \quad c^2 + 2ab\cos C = a^2 + b^2$$

$$\Rightarrow \quad 2ab\cos C = a^2 + b^2 - c^2$$

$$\Rightarrow \quad \boxed{\cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

This arrangement (and the corresponding formulae for cos A or cos B) will enable you to find any angle of a triangle given all three sides.

What will the formulas be for cos A or cos B?

Unlike the sine rule, there is no possible ambiguity since, if C is obtuse, cos C will turn out to be negative rather than positive. (So with the cosine rule you can trust the inverse cosine function on your calculator!)

Example

Find all three angles of triangle LMN given $l = 72$, $m = 38$ m and $n = 49$ m.

Solution

To find L, use

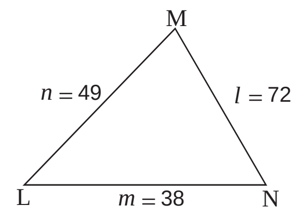
$$\cos L = \frac{m^2 + n^2 - l^2}{2mn}$$

$$= \frac{38^2 + 49^2 - 72^2}{2 \times 38 \times 49}$$

$$= \frac{-1339}{3724} = -0.359560.$$

$\Rightarrow \quad L = 111.1°$.

Having found one angle, the next step could be to use either the cosine rule again or the sine rule.



$$\cos M = \frac{l^2 + n^2 - m^2}{2ln}$$

$$= \frac{72^2 + 49^2 - 38^2}{2 \times 72 \times 49} = 0.859410$$

$\Rightarrow \quad M = 30.7°$

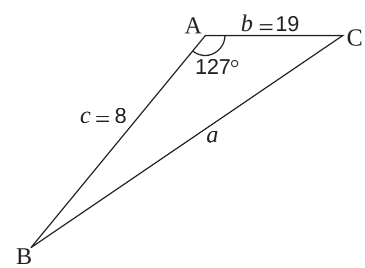
$\Rightarrow \quad N = 180° - 111.1° - 30.7°$

$= 38.2°$.

Example

In triangle ABC, $b = 19$ m, $c = 8$ m and $A = 127°$.

Find a and angles B and C.



Solution

Using the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 19^2 + 8^2 - 2 \times 19 \times 8\cos 127°$$
$$= 425 - 304 \times (-0.601\ 815)$$
$$= 425 + 182.95176$$
$$= 607.95176$$
$$\Rightarrow \quad a = 24.7 \text{ m}$$

Next, the sine rule can be used to find B and C.

Now $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\Rightarrow \quad \frac{24.7}{\sin 127°} = \frac{19}{\sin B}$$

$$\Rightarrow \quad \sin B = \frac{19\sin 127°}{24.7} = 0.614335$$

$$\Rightarrow \quad B = 37.9°$$

and $C = 180° - 127° - 38° = 15°$ to the nearest degree.

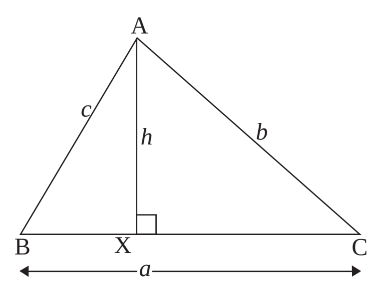
*Exercise 15I

Solve the following triangles given the relevant information :

1. In triangle ABC: $a = 18$ cm, $b = 13$ cm, $c = 8$ cm.
2. In triangle DEF: $D = 13.8°$, $e = 9.2$ m, $f = 13.4$ m.
3. In triangle LMN: $l = 33$ mm, $m = 20$ mm, $N = 71°$.
4. In triangle XYZ: $x = 4$ m, $y = 7$ m, $z = 9.5$ m.
5. In triangle PQR: $p = 9$ cm, $q = 40$ cm, $r = 41$ cm.
6. In triangle UVW: $U = 37°$, $u = 88.3$ m, $w = 97$ m.

*15.10 Area of a triangle

The approach adopted in obtaining the sine rule gives an easy way of finding a formula for the area of any triangle.



With base a, the height $h = b\sin C$ [or $c\sin B$] and the area of a triangle is given by

$$\text{area} = \frac{1}{2}\text{ base} \times \text{height}$$

$$= \frac{1}{2} \times a \times b \sin C$$

$$\Rightarrow \boxed{\text{area} = \frac{1}{2} ab \sin C}$$

$$= \frac{1}{2} ac \sin B$$

$$= \frac{1}{2} bc \sin A$$

depending upon the choice of base.

Notice again that in each case the formula requires any two sides and the included angle.

*Exercise 15J

Find the areas of the triangles in Exercise 15I, Questions 1 to 6.

15.11 Miscellaneous Exercises

1. Simplify the following expressions :
   (a) $\cos 37° \cos 23° - \sin 37° \sin 23°$
   (b) $\sin 28° \cos 42° + \cos 28° \cos 48°$
   (c) $\sin\left(\frac{\pi}{3} + x\right) - \sin\left(\frac{\pi}{3} - x\right)$
   (d) $\cos\left(\frac{\pi}{6} + x\right) + \cos\left(\frac{\pi}{6} - x\right)$
   (e) $\sin\left(\frac{2\pi}{3} - x\right) + \sin\left(x - \frac{\pi}{3}\right)$
2. Prove the following identities :
   (a) $\frac{\sin\theta + \sin 3\theta + \sin 5\theta}{\cos\theta + \cos 3\theta + \cos 5\theta} = \tan 3\theta$
   (b) $\frac{\sin(x-y) + \sin x + \sin(x+y)}{\cos(x-y) + \cos x + \cos(x+y)} = \tan x$
   (c) $\cos(A+B+C) + \cos(A+B-C) + \cos(A-B+C)$
   $+\cos(-A+B+C) = 4\cos A \cos B \cos C$
3. Solve the equation $4\tan^2 x + 12\sec x + 1 = 0$ giving all solutions in degrees, to the nearest degree, in the interval $-180° < x < 180°$. (AEB)
4. Solve the equation $\sqrt{3}\tan\theta - \sec\theta = 1$ giving all solutions in the interval $0° < \theta < 360°$. (AEB)
5. Prove the identity $\sin 3A = 3\sin A - 4\sin^3 A$. Hence show that $\sin 10°$ is a root of the equation $8x^3 - 6x + 1 = 0$. (AEB)
6. Prove the identity $\tan\theta + \cot\theta = 2\operatorname{cosec} 2\theta$. Find, in radians, all the solutions of the equation $\tan x + \cot x = 8\cos 2x$ in the interval $0 < x < \pi$. (AEB)

*7. Find the area of triangle ABC given
   (a) $a = 42$ cm, $b = 31$ cm, $C = 58.1°$
   (b) $A = 17.6°$, $b = 127$ m, $c = 98$ m

*8. A triangle has sides of length 2 cm, 3 cm and 4 cm. Find the area of the triangle.

*9. In triangle PQR, $R = 42.5°$, $p = 9$ m and $q = 12.2$ m. Find the area of the triangle and the length of the third side, r.
   Deduce the distance of R from the line PQ.

*10. In triangle XYZ, $x = 17$, $y = 28$ and angle $X = 34°$. Find the length of the remaining side and the size of angles Y and Z.

*11. Solve triangle PQR given

(a) $P = 14.8°$, $R = 59.1°$, $r = 87$ m.

(b) $P = 67°$, $p = 73$ m, $q = 42$ m.

(c) $p = 22$ cm, $q = 89$ cm, $r = 100$ cm.

*12. $a = 27$ m, $b = 32$ m and $c = 30.6$ m in triangle ABC. Find the smallest angle of the triangle, and its area.