14 FURTHER CALCULUS

Objectives

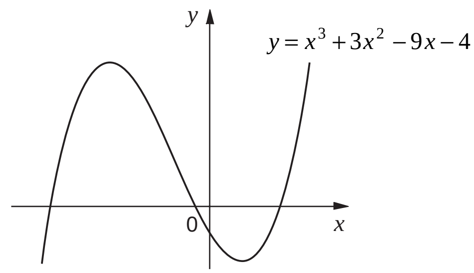
After studying this chapter you should

- be able to find the second derivatives of functions;
- be able to use calculus to find maximum or minimum of functions;
- be able to differentiate composite functions or 'function of a function';
- be able to differentiate the product or quotient of two functions;
- be able to use calculus in curve sketching.

14.0 Introduction

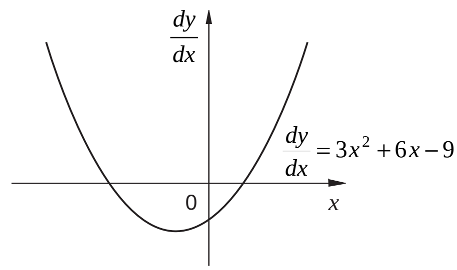
You should have already covered the material in Chapters 8, 11 and 12 before starting this chapter. By now, you will already have met the ideas of calculus, both differentiation and integration, and you will have used the techniques developed in the earlier chapters to solve problems. This section extends the range of problems you can solve, including finding the greatest or least value of a function and differentiating complicated functions.

14.1 Rate of change of the gradient



The sketch opposite shows the graph of

$$y = x^3 + 3x^2 - 9x - 4 .$$



You have already seen that the derivative of this function is given by

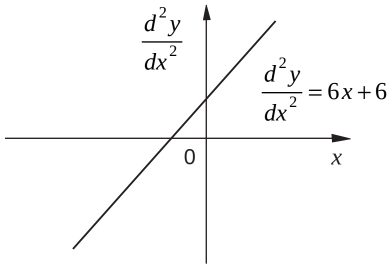
$$\frac{dy}{dx} = 3x^2 + 6x - 9 .$$

This is also illustrated opposite.

The gradient function, $\frac{dy}{dx}$, is also a function of x, and can be differentiated again to give the **second** differential

$$\frac{d^2y}{dx^2} = 6x + 6.$$

Again this is illustrated opposite.



Exercise 14A

1. Find the second derivative of these functions :

   (a) $y = x^3$ (b) $y = x^4$ (c) $y = x^2$

   (d) $y = x$ (e) $y = \frac{1}{x}$ (f) $y = 4x^3 - 12x^2 + 5$

   (g) $y = 3x + 1$ (h) $y = e^x$ (i) $y = \ln x$

2. Find $\frac{d^3y}{dx^3}$ if y is

   (a) x^4 (b) $5x^2 + \frac{3}{x^2}$ (c) e^x

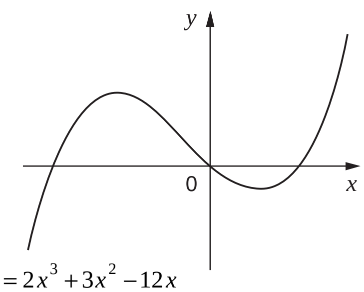
   (d) $\ln x$ (e) $2x$

14.2 Stationary points

In this section you will consider the curve with equation

$$y = 2x^3 + 3x^2 - 12x.$$

This is cubic, and a rough sketch of its graph is shown opposite. It has two **stationary points (**sometimes called **turning points**) at which the gradient is zero.



How can you find the coordinates of the stationary points?

Activity 1

Find the coordinates of the stationary points for the curve with equation

$$y = 2x^3 + 3x^2 - 12x.$$

In Chapter 8 you saw that the nature of the stationary points can be determined by looking at the gradient on each side of the stationary point. Here an alternative more formal method is developed, based on using second derivatives.

Activity 2

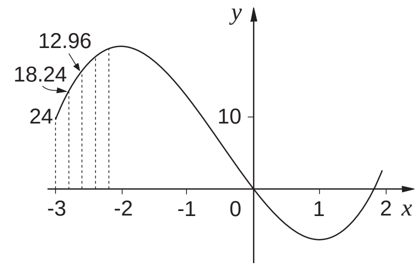
Draw an accurate sketch of the curve with equation

$$y = 2x^3 + 3x^2 - 12x$$

between $x = -3$ and $+2$. Choose the y axis to show values between -10 and $+20$.

For every x value –3, –2.8, –2.6, –2.4, ..., 2, note the gradient on the diagram.

Plot a graph of the gradient function and note how it behaves near the stationary point of the function.



For a **maximum** value of a function, note that the gradient is decreasing in value as it passes through the value zero at the stationary point, whereas for a **minimum** value this gradient is increasing.

The result can be summarised as

For **stationary points** of a function $y(x)$

$$\frac{dy}{dx} = 0\,.$$

If $\dfrac{d^2y}{dx^2} < 0$ at a stationary point, it corresponds to a **maximum** value of y.

If $\dfrac{d^2y}{dx^2} > 0$ at a stationary point, it corresponds to a **minimum** value of y.

Example

Find maxima and minima of the curve with equation

$$y = \frac{1}{4}x^4 + \frac{1}{3}x^3 - 6x^2 + 3.$$

Hence sketch the curve.

Solution

For stationary points, $\frac{dy}{dx} = 0$, which gives

$$\frac{dy}{dx} = x^3 + x^2 - 12x.$$

So $$\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x^3 + x^2 - 12x = 0$$

$$x(x^2 + x - 12) = 0$$

$$x(x+4)(x-3) = 0.$$

Hence there are stationary points at $x = 0, -4$ and 3. To find out their nature, the second derivative is used. Now

$$\frac{d^2y}{dx^2} = 3x^2 + 2x - 12.$$

At $x = 0$, $\frac{d^2y}{dx^2} = -12 < 0$

$\Rightarrow$ a maximum at $x = 0$ of value $y = 3$.

At $x = -4$, $\frac{d^2y}{dx^2} = 3(-4)^2 + 2(-4) - 12 = 28 > 0$

$\Rightarrow$ minimum at $x = -4$ of value

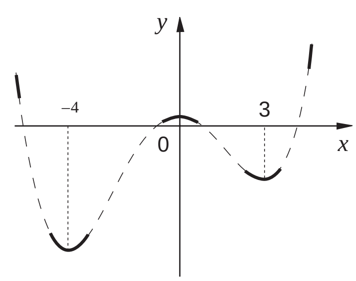
$$y = \frac{1}{4}(-4)^4 + \frac{1}{3}(-4)^3 - 6(-4)^2 + 3 = -\frac{151}{3}.$$

At $x = 3$, $\frac{d^2y}{dx^2} = 3(3)^2 + 2(3) - 12 = 21 > 0$

$\Rightarrow$ minimum at $x = 3$ of value

$$y = \frac{1}{4}(3)^4 + \frac{1}{3}(3)^3 - 6(3)^2 + 3 = -\frac{87}{4}.$$

The information so far found can be sketched on a graph (not to scale). You also know that as $x \to \pm\infty, y \to +\infty$. It is now clear how to sketch the shape - shown dashed on the diagram.



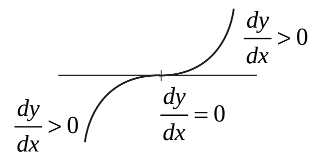
There is one further type of stationary point to be considered and that is a **point of inflection**. An example is given in the next activity.

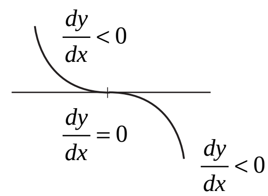
Activity 3

Find the stationary point of $y = x^3$.

What is the value of $\frac{d^2y}{dx^2}$ at the stationary point?

Sketch the graph of $y = x^3$.





For a **horizontal** point of inflection, not only does $\frac{dy}{dx} = 0$, but also $\frac{d^2y}{dx^2} = 0$, and $\frac{d^3y}{dx^3} \neq 0$, at the point. These are **sufficient** but not **necessary** conditions, as can be seen by considering $y = x^5$.

Activity 4

Find the stationary point of $y = x^5$.

What is the value of $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ at its stationary point?

Sketch the graph of $y = x^5$.

Example

Find the nature of the stationary points of the curve with equation

$$y = x^4 + 4x^3 - 6$$

Sketch a graph of the curve.

Solution

Now

$$\frac{dy}{dx} = 4x^3 + 12x^2$$

$$= 0 \text{ when } 4x^3 + 12x^2 = 0$$

$$\Rightarrow \quad 4x^2(x+3) = 0$$

$$\Rightarrow \quad x = 0, -3 \text{ for stationary points}$$

But $$\frac{d^2y}{dx^2} = 12x^2 + 24x$$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$, but $\frac{d^3y}{dx^3} = 24x + 24 = 24 > 0$ at $x = 0$.

So there is a point of inflection at (0, – 6).

$$\text{At } x = -3, \frac{d^2y}{dx^2} = 12(-3)^2 + 24(-3)$$
$$= 108 - 72 = 36 > 0$$

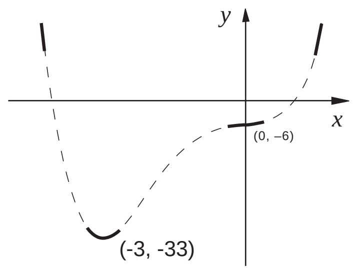
So there is a minimum at $x = -3$ of value

$$y(-3) = (-3)^4 + 4(-3)^3 - 6$$
$$= 81 - 4 \times 27 - 6$$
$$= -33$$

To sketch the curve, also note that

$$y \to \infty \text{ as } x \to \pm\infty.$$

You can then deduce the form of curve, shown dashed opposite.



Finally, in this section, it should be noted that the conditions given earlier are **sufficient** conditions to guarantee max/min values, but not **necessary**.

Activity 5

Draw the graph of $y = x^4$.

Does it have any maximum or minimum values?

Do the conditions hold?

By now you are probably getting confused since there is a rule to determine the nature of stationary points, yet not all functions satisfy it. So let it be stressed that

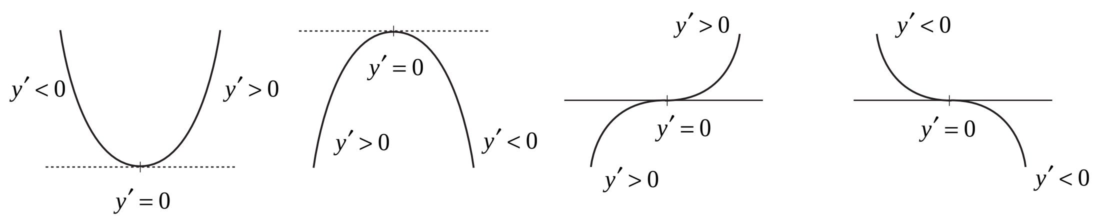
(a) Stationary points are always given by $\frac{dy}{dx} = 0$.

(b) If, at the stationary point, $\frac{d^2y}{dx^2} > 0$, there is a minimum,

whereas if $\frac{d^2y}{dx^2} < 0$, there is a maximum.

(c) If, at the stationary point, $\frac{d^2y}{dx^2} = 0$, then there is a point of inflection provided $\frac{d^3y}{dx^3} \neq 0$.

Conditions (b) and (c) are sufficient to guarantee the nature of the stationary point, but, as you have already seen **not** necessary. When this analysis does not hold, that is when $\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$ at a stationary point, it is easier to consider the **sign** of the gradient each side of the stationary point.



gradients

< ⋮ 0 ⋮ >	> ⋮ 0 ⋮ <	> ⋮ 0 ⋮ >	< ⋮ 0 ⋮ <
minimum	maximum	point of inflection	point of inflection

Example

Show that $y = x^4$ has a minimum at $x = 0$.

Solution

For stationary points $\frac{dy}{dx} = 4x^3$

$$= 0 \Rightarrow x = 0$$

But, if $x = 0$, $\frac{d^2y}{dx^2} = 12x^2 = 0$

and $\frac{d^3y}{dx^3} = 24x = 0$

So you cannot use the usual results, but at, for example,

$$x = -0.1 \Rightarrow \frac{dy}{dx} = 4(-0.1)^3 = -0.004 < 0$$

$$x = 0.1 \Rightarrow \frac{dy}{dx} = 4(0.1)^3 = 0.004 > 0$$

So there is a **minimum** at $x = 0$.

Exercise 14B

1. Find the turning points of these curves using differentiation. In each case, find out whether the points found are maxima, minima or points of inflection.

   (a) $y = 2x^2 + 3x - 1$ (b) $y = x^3 - 12x + 6$

   (c) $y = x^3 + 2x^2 - 5x - 6$ (d) $y = x^2 - 4x + 2\ln(x)$

   (e) $y = e^x - 4x$ (f) $y = e^x + 5$

2. Investigate the nature of the stationary points of the curve

   $$y = (x-1)(x-a)^2$$

   when (a) $a > 1$ (b) $0 < a < 1$. In each case give a sketch of the curve.

14.3 Differentiating composite functions

The crucial result needed here is the 'function of a function' rule. It can be justified by noting that if y is a function of u, and u is a function of x

i.e. $y = f(u)$ and $u = g(x)$

then a small change, δx say, in x will result in a small change, say δu in u, which in turn results in a small change, δy in y. Now

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

and if you let $\delta x \to 0$, then $\delta u \to 0$ and $\delta y \to 0$, and the equation becomes

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

For example, suppose

$$y = (3x+2)^2$$

then one way to differentiate this function is to multiply out the brackets and differentiate term by term;

$$y = (3x+2)(3x+2)$$

$$= 9x^2 + 12x + 4$$

and $\dfrac{dy}{dx} = 18x + 12.$

Another method, which will be even more useful as the functions get more complicated, is to introduce a new variable u defined by

$$u = 3x + 2$$

so that $y = u^2$.

Now $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = 3$, so using the result above

$$\frac{dy}{dx} = (2u) \times 3 = 6(3x+2),$$ as before.

Activity 6

Use both methods described above to differentiate the functions:

(a) $y = (5x-1)^2$ (b) $y = (3-2x)^3$.

Example

If $y = e^{x^2}$, find $\frac{dy}{dx}$.

Solution

As before, introduce a new variable u defined by

$$u = x^2$$

so that

$$y = e^u.$$

Now $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x$

giving $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (e^u)2x = 2xe^{x^2}$.

What is the value of $\int xe^{x^2}\,dx$?

Another important result needed is given by

$$\boxed{\frac{dy}{dx} = 1 / \frac{dx}{dy}}$$

but note that this is only true for **first** derivatives.

For example, if $y = x^2$

then $\quad \dfrac{dy}{dx} = 2x$

But, expressing x as a function of y,

$$x = y^{\frac{1}{2}}$$

and $\quad \dfrac{dx}{dy} = \dfrac{1}{2} y^{-\frac{1}{2}} = \dfrac{1}{2y^{\frac{1}{2}}} = \dfrac{1}{2x}$ (generalising the result that $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$ for any n)

So, as expected $\dfrac{dy}{dx} = 1 / \dfrac{dx}{dy}$

Activity 7

Verify the result $\dfrac{dy}{dx} = 1 / \dfrac{dx}{dy}$ when

(a) $y = x^3$ (b) $y = \frac{1}{2}(x+3)$

The proof of the result is based on using small increments (increase) δx and δy, noting that

$$\frac{\delta y}{\delta x} = 1 / \frac{\delta x}{\delta y}$$

and taking the limit as $\delta x \to 0$ (and $\delta y \to 0$).

This section is concluded by using this result in finding the derivative of the function

$$y = a^x$$

To achieve this, you must first take 'logs' of both sides to give

$$\ln y = \ln(a^x) = x \ln a \text{ (properties of logs)}$$

Defining $\quad u = (\ln a)x \Rightarrow \dfrac{du}{dx} = \ln a$

and $\quad u = \ln y$

Hence $\frac{du}{dy} = \frac{1}{y}$

giving $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

$= \left(1/\frac{du}{dy}\right)\ln a$

$= y\ln a$

$\frac{d}{dx}(a^x) = (\ln a)a^x$

so, for example

$\frac{d}{dx}(2^x) = (\ln 2)2^x$

What happens if $a = e$ in the formula above?

Exercise 14C

1. Differentiate these functions :

   (a) $y = (2x-5)^8$ $\quad (u = 2x-5)$

   (b) $y = (x^2+x^3)^{10}$ $\quad (u = x^2+x^3)$

   (c) $y = \frac{1}{x-2}$ $\quad (u = x-2, y = u^{-1})$

   (d) $y = \frac{1}{(3x+1)^2}$ $\quad (u = 3x+1, y = u^{-2})$

   (e) $y = \sqrt{x+1}$ $\quad (u = x+1, y = u^{½})$

   (f) $y = (e^x - x)^6$ $\quad (u = e^x - x)$

   (g) $y = \ln(3x+4)$ $\quad (u = 3x+4)$

   (h) $y = e^{\sqrt{x}}$ $\quad (u = -\sqrt{x})$

2. Find any turning points on these curves :

   (a) $y = e^{x^2}$

   (b) $y = e^{-(x+4)^2}$ $\quad \left(u = -(x+4)^2\right)$

   (c) $y = (\ln(x))^2$ $\quad (u = \ln(x))$

   (d) $y = \left(x + \frac{1}{x}\right)^4$ $\quad \left(u = x + \frac{1}{x}\right)$

3. A kettle contains water which is cooling according to the equation $W = 80e^{-0.04t} + 20$ where W is the temperature of the water in °C at time t (in minutes) after the kettle was switched off. Find the rate at which the water is cooling in °C/min when $t = 30$ and when $t = 60$.

4. The population of a country is modelled by the function

   $P = 12 \times (1.03)^t$

   where P is the population in millions and t is the time in years after the start of 1990. Find the rate at which the population is increasing in millions per year at the start of the year 2000.

5. (a) Differentiate e^{2x}. Hence write down $\int e^{2x}dx$.

   (b) Differentiate e^{-x}, and so write down $\int e^{-x}dx$.

6. Differentiate e^{x^2}, and use your result to find $\int xe^{x^2}dx$.

7. The derivative of $\ln x$ is $\frac{1}{x}$.

   By differentiating a suitable logarithmic function, find $\int \frac{1}{x+2}dx$.

14.4 Integration again

The results that have been developed in the last section are, as you will see, very useful in integration. For example, if

$$y = (x+5)^4$$

then $\dfrac{dy}{dx} = 4(x+5)^3$ (using the methods in the last section).

Hence $\int 4(x+5)^3\,dx = (x+5)^4 + C$ (C is arbitrary constant)

or $\int (x+5)^3\,dx = \dfrac{1}{4}(x+5)^4 + C'$ $\quad \left(C' = \dfrac{1}{4}C\right)$.

Activity 8

Differentiate $y = (ax+b)^{n+1}$ and hence deduce the value of

$$\int (ax+b)^n\,dx.$$

Use your results to find:

(a) $\int (3x-1)^9\,dx$ (b) $\int \dfrac{1}{(2x+4)^2}\,dx$ (c) $\int \sqrt{x-1}\,dx$.

Similarly, it can be seen that if

$$y = \ln(ax+b)$$

then $\dfrac{dy}{dx} = \dfrac{a}{(ax+b)}$.

Hence $\int \dfrac{a\,dx}{(ax+b)} = \ln(ax+b) + C$

or $\boxed{\int \dfrac{dx}{(ax+b)} = \dfrac{1}{a}\ln(ax+b) + C'}$ $\qquad C' = \dfrac{1}{a}C$

Activity 9

Use the result above to evaluate:

(a) $\int \dfrac{dx}{(x+1)}$ (b) $\int \dfrac{dx}{4-x}$ (c) $\int \dfrac{dx}{1-2x}$.

Another important method of integration is based on the result that if

$$y = \ln f(x)$$

then defining $u = f(x), y = \ln u$ and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times f'(x)$$

$$= \frac{f'(x)}{f(x)}$$

Hence
$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \ln f(x) + C}$$

Example

Find $\displaystyle\int \frac{(3x^2 + x)}{(2x^3 + x^2)}\, dx$

Solution

If $f(x) = 2x^3 + x^2$, then $f'(x) = 6x^2 + 2x$, so the integral can be written as

$$I = \int \frac{(3x^2 + x)}{(2x^3 + x^2)}\, dx$$

$$= \frac{1}{2}\int \frac{2(3x^2 + x)}{(2x^3 + x^2)}\, dx$$

$$= \frac{1}{2}\int \frac{f'(x)}{f(x)}\, dx \qquad (\text{when } f(x) = 2x^3 + x^2)$$

$$= \frac{1}{2}\int \ln f(x) + C$$

$$= \frac{1}{2}\ln(6x^2 + 2x) + C$$

Exercise 14D

1. Find

(a) $\int (2x+1)^4\,dx$ (b) $\int \frac{1}{(x-5)^2}\,dx$

(c) $\int \frac{1}{\sqrt{x+1}}\,dx$ (d) $\int \sqrt{4x-1}\,dx$

2. Evaluate

(a) $\int_0^1 \frac{dx}{(x+1)}$ (b) $\int_1^2 \frac{dx}{(2x-1)}$

3. Find

(a) $\int \frac{dx}{(3x+2)}$ (b) $\int \frac{x}{x^2+1}\,dx$

(c) $\int \frac{e^x}{1+e^x}\,dx$

4. Evaluate $\int_0^1 xe^{-x^2}\,dx$

14.5 Differentiating products and quotients

These are very useful formulae for differentiating both products and quotients. For example, if y is defined as the product of the functions, u and v of x, then

$$y(x) = u(x)\,v(x)$$

Using the basic definition of a derivative, let δx be a small change in x, then consider

$$\begin{aligned} y(x+\delta x) - y(x) &= u(x+\delta x)\,v(x+\delta x) - u(x)\,v(x) \\ &= [u(x+\delta x) - u(x)]\,v(x+\delta x) \\ &\quad + u(x)[v(x+\delta x) - v(x)] \end{aligned}$$

(the middle two terms cancel)

Diving both sides by δx gives

$$\frac{y(x+\delta x) - y(x)}{\delta x} = \frac{(u(x+\delta x) - u(x))}{\delta x}\,v(x+\delta x) + u(x)\frac{(v(x+\delta x) - v(x))}{\delta x}$$

and taking the limit as $\delta x \to 0$, gives

$$\boxed{\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}}$$

(since $v(x+\delta x) \to v(x)$ as $\delta x \to 0$)

Example

If $y = x^2(x-1)^2$, find $\dfrac{dy}{dx}$ by

(a) using the formula above;
(b) multiplying out and differentiating term by term.

Solution

(a) Here $u = x^2, v = (x-1)^2$,

so $$\frac{dy}{dx} = 2x(x-1)^2 + x^2\,2(x-1)$$

$$= 2x(x-1)(x-1+x)$$

$$= 2x(x-1)(2x-1).$$

(b) $y = x^2(x^2-2x+1) = x^4 - 2x^3 + x^2$.

$$\frac{dy}{dx} = 4x^3 - 6x^2 + 2x$$

$$= 2x(2x^2 - 3x + 1)$$

$$= 2x(x-1)(2x-1) \quad \text{(as before).}$$

Example

If $y = (x+1)e^{-x}$, find $\dfrac{dy}{dx}$.

Solution

This time, you must use the formula, with $u = x+1, v = e^{-x}$.

So $$\frac{dy}{dx} = 1 \times \left(e^{-x}\right) + (x+1)\left(-e^{-x}\right)$$

$$= -xe^{-x}.$$

Turning to the equivalent formula for a quotient, with

$$y = \frac{u}{v} = u\left(\frac{1}{v}\right)$$

which is the product of u and $\dfrac{1}{v}$.

Now
$$\frac{d}{dx}\left(\frac{1}{v}\right)=\frac{d}{dv}\left(\frac{1}{v}\right)\frac{dv}{dx} \quad \text{('function of a function')}$$
$$=-\frac{1}{v^2}\frac{dv}{dx}$$

and
$$\frac{dy}{dx} = \frac{du}{dx}\times\frac{1}{v} + u\frac{d}{dx}\left(\frac{1}{v}\right)$$
$$= \frac{du}{dx}\times\frac{1}{v} + u\left(-\frac{1}{v^2}\frac{dv}{dx}\right).$$

i.e.
$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\left(v\frac{du}{dx}-u\frac{dv}{dx}\right)}{v^2}}$$

Example

Use the quotient formula to find $\frac{d}{dx}((1+x)e^{-x})$.

Solution

Here $u=(1+x), v=e^x$, so that

$$y=\frac{u}{v}=\frac{(1+x)}{(e^x)}=(1+x)e^{-x} \ .$$

Thus $\frac{dy}{dx}=\frac{e^x\times 1-(1+x)e^x}{(e^x)^2}$, since $\frac{d}{dx}(e^x)=e^x$

$$=-\frac{xe^x}{(e^x)^2}$$
$$=-\frac{x}{e^x}$$
$$=-xe^{-x} \qquad \text{(as in the previous example).}$$

So you can see that the quotient formula is just another form of the product formula. It is though sometimes very convenient to use.

Example

Differentiate $y = \dfrac{x-1}{2x-3}$ with respect to x.

Solution

Here $u = x-1$, $v = 2x-3$ and using the quotient formula

$$\frac{dy}{dx} = \frac{1\times(2x-3) - 2(x-1)}{(2x-3)^2}$$

$$= \frac{2x-3-2x+2}{(2x-3)^2}$$

$$= \frac{-1}{(2x-3)^2}.$$

*Activity 10

Develop a formula for differentiating the product of **three** functions .

i.e. $y(x) = u(x)\,v(x)\,w(x)$.

Example

Find any stationary points for the curve with equation

$$y = \frac{x^3}{(1+x)}.$$

Sketch the curve.

Solution

Here $u = x^3$, $v = 1+x$, so $y = \dfrac{u}{v}$ and using the quotient formula

$$\frac{dy}{dx} = \frac{3x^2(1+x) - x^3\times 1}{(1+x)^2}$$

$$= \frac{3x^2+2x^3}{(1+x)^2}.$$

For stationary points,

$$3x^2 + 2x^3 = 0$$

$$x^2(3+2x) = 0$$

$$\Rightarrow \quad x = 0, -\tfrac{3}{2}$$

To determine the nature of these stationary points, you can check the sign of $\frac{dy}{dx}$ each side of the points.

So, for $x = 0$

$$\frac{dy}{dx}(-0.1) = \frac{3(-0.1)^2 + 2(-0.1)^3}{(1-0.1)^2} = \frac{0.03 - 0.002}{(0.9)^2}$$

$$= \frac{0.028}{0.81} > 0$$

and

$$\frac{dy}{dx}(0.1) = \frac{3(0.1)^2 + 2(0.1)^3}{(1+0.1)^2} = \frac{0.03 + 0.002}{(1.1)^2}$$

$$= \frac{0.032}{1.21} > 0$$

So there is a point of inflection at $x = 0$ and value of function here is $y = 0$.

For $x = -1.5$,

$$\frac{dy}{dx}(-1.6) = \frac{3(-1.6)^2 + 2(-1.6)^3}{(1-1.6)^2} = \frac{-0.512}{0.36} < 0$$

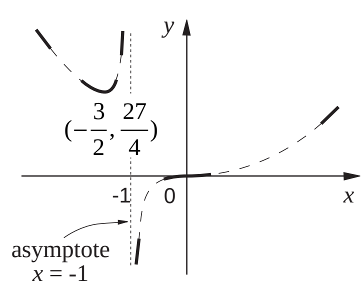
$$\frac{dy}{dx}(-1.4) = \frac{3(-1.4)^2 + 2(-1.4)^3}{(1-1.4)^2} = \frac{0.392}{0.16} > 0$$

Hence there is a minimum at $x = -1.5$ of value $y = \frac{27}{4}$.

Also note that the function has an **asymptote** at $x = -1$, that is

$$y \to \pm\infty \text{ as } x \to -1.$$

As $x \to \pm\infty$, $y = \frac{x^3}{1+x} \approx \frac{x^3}{x} = x^2$



All this information is marked on the diagram opposite, and the curve is sketched in dotted.

Exercise 14E

1. Differentiate the following functions with respect to x :

   (a) $y = x e^x$ (b) $y = x^2 \ln(x)$

   (c) $y = \dfrac{2x+1}{x^2-3}$ (d) $y = \dfrac{e^x}{1+x}$

2. Find any stationary points on the curves in Question 1.

3. Find any stationary points on the curve

$$y = \frac{\ln(\sqrt{x}+1)}{e^x}$$

4. Differentiate $\sqrt{\dfrac{x+1}{2x-1}}$

5. Differentiate $\sqrt{e^x(1+x)}$

6. Differentiate $\dfrac{\ln(x-2)}{\sqrt{x}}$

7. Sketch the curve with equation

$$y = x^3 + \frac{3}{2}x^2 - 6x$$

   by first finding all stationary points and their nature.

8. Sketch the curve with equation

$$y = \frac{1}{x^2}e^{-\frac{1}{x^2}}$$

9. Find any stationary points of the curve with equation

$$y = \frac{x^2}{1-x}$$

   Hence sketch the curve, indicating any asymptotes.

14.6 Miscellaneous Exercises

1. Find the maxima and/or minima of these functions :

   (a) $y = x^2 - 5x + 6$ (b) $y = 3x - 2x^2 + 8$

   (c) $y = x^3 + 2x^2 + x - 4$ (d) $y = 6x^2 - 2x^3 + 48x$

   (e) $y = \dfrac{e^x}{x}$

2. Differentiate these functions with respect to x

   (a) $(5-3x)^8$ (b) $\dfrac{1}{\left(\sqrt{x}+1\right)}$ (c) 3^x

   (d) $\sqrt{\dfrac{1+x}{1-x}}$ (e) $\ln(2^x)$ (f) $e^{-x^{-2}}$

3. Differentiate e^{3x-1}. Hence write down $\int e^{3x-1}dx$

4. Find $\int (2x-4)^5\,dx$.

5. Calculate $\int_3^6 \dfrac{1}{3x-1}dx$.

6. Differentiate $\ln(x^2+1)$. Hence find $\int_0^1 \dfrac{x}{x^2+1}dx$.