13 SEQUENCES AND SERIES

Objectives

After studying this chapter you should

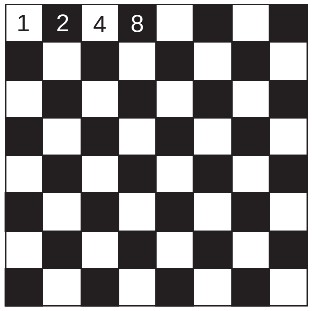
- be able to recognise geometric and arithmetic sequences;
- understand $\sum$ notation for sums of series;
- be familiar with the standard formulas for $\sum r$, $\sum r^2$ and $\sum r^3$;

13.0 Introduction

Suppose you go on a sponsored walk. In the first hour you walk 3 miles, in the second hour 2 miles and in each succeeding hour $\frac{2}{3}$ of the distance the hour before. How far would you walk in 10 hours? How far would you go if you kept on like this for ever?

This gives a sequence of numbers: 3, 2, $1\frac{1}{3}$, .. etc. This chapter is about how to tackle problems that involve sequences like this and gives further examples of where they might arise. It also examines sequences and series in general, quick methods of writing them down, and techniques for investigating their behaviour.

Number of grains of corn shown



Legend has it that the inventor of the game called **chess** was told to name his own reward. His reply was along these lines.

'Imagine a chessboard.

Suppose 1 grain of corn is placed on the first square,

2 grains on the second,
4 grains on the third,
8 grains on the fourth,

and so on, doubling each time up to and including the 64th square. I would like as many grains of corn as the chessboard now carries.'

It took his patron a little time to appreciate the enormity of this request, but not as long as the inventor would have taken to use all the corn up.

Activity 1

(a) How many grains would there be on the 64th square?

(b) How many would there be on the *n*th square?

(c) Work out the numerical values of the first 10 terms of the sequence.

$$2^0,\ 2^0+2^1,\ 2^0+2^1+2^2 \text{ etc.}$$

(d) How many grains are there on the chessboard?

13.1 Geometric sequences

The series of numbers 1, 2, 4, 8, 16 ... is an example of a **geometric sequence** (sometimes called a geometric progression). Each term in the progression is found by **multiplying** the previous number by 2.

Such sequences occur in many situations; the multiplying factor does not have to be 2. For example, if you invested £2000 in an account with a fixed interest rate of 8% p.a. then the amounts of money in the account after 1 year, 2 years, 3 years etc. would be as shown in the table. The first number in the sequence is 2000 and each successive number is found by multiplying by 1.08 each time.

Number of years	Money in account (£)
0	2000
1	2160
2	2332.80
3	2159.42
4	2720.98

Accountants often work out the residual value of a piece of equipment by assuming a fixed depreciation rate. Suppose a piece of equipment was originally worth £35 000 and depreciates in value by 10% each year. Then the values at the beginning of each succeeding year are as shown in the table opposite. Notice that they too form a geometric progression.

Year	Value (£)
0	35 000
1	31 500
2	28 350
3	25 515
4	22 963.50

The chessboard problem in Activity 1 involved adding up

$$2^0+2^1+2^2+\ldots+2^{63}.$$

The sum of several terms of a sequence is called a **series**. Hence the sum $2^0+2^1+2^2+\ldots+2^{63}$ is called a **geometric series** (sometimes geometric progression, GP for short)

Activity 2 Summing a GP

In Activity 1 you might have found a formula for

$$1+2+2^2+\ldots+2^{n-1}.$$

(a) Work out the values of

$$3^0,\ 3^0+3^1,\ 3^0+3^1+3^2$$

(b) Find a formula for

$$1 + 3 + 3^2 + \ldots + 3^{n-1}$$

(c) Find a formula for $1 + 4 + 4^2 \ \ldots + \ 4^{n-1}$

(d) Now find a formula for

$$1 + r + r^2 + \ldots + r^{n-1}$$

where r is any number. Test your theory.

(e) In practice, geometric series do not always start with 1. Suppose the first term is a. How is the series in part (d) altered? How can you adapt your formula for the total of all terms?

The general form of a geometric sequence with n terms is

$$a, ar, ar^2, \ldots, ar^{n-1}$$

The ratio r of consecutive terms, is known as the **common ratio.** Notice that the nth term of the sequence is ar^{n-1}.

In the chessboard problem the solution involved adding up the first 64 terms. The sum of the first n terms of a series is often denoted by S_n, and there is a formula for S_n which you may have found in Activity 2. Here is a way of proving the formula, when $r \neq 1$.

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad (1)$$

Multiply both sides by r :

$$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \qquad (2)$$

Notice that the expressions for S_n and rS_n are identical, with the exception of the terms a and ar^n. Subtracting equation (1) from equation (2) gives

$$rS_n - S_n = ar^n - a$$

$$\Rightarrow \qquad S_n(r-1) = a\left(r^n - 1\right)$$

$$\Rightarrow \qquad S_n = \frac{a\left(r^n - 1\right)}{r-1}$$

Activity 3 Understanding and using the formula

(a) Sometimes it is useful to write

$$S_n = \frac{a(1-r^n)}{1-r} \text{ instead of } S_n = \frac{a(r^n-1)}{r-1}$$

Why are these formulae identical? When might it be more convenient to use the alternative form?

(b) For what value of r do these formulas not hold? What is S_n in this case?

Example:

Find

(a) $4+6+9+ \ldots + 4\times(1.5)^{10}$

(b) $8+6+4.5+ \ldots + 8\times(0.75)^{25}$

Solution

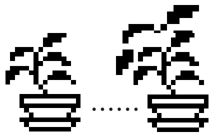
(a) First term $a=4$, common ratio $r=1.5$, number of terms $n=11$;

$$S_{11} = \frac{4(1.5^{11}-1)}{1.5-1} = 684.0 \text{ to } 4 \text{ s.f.}$$

(b) First term $a=8$, common ratio $r=0.75$, number of terms $n=26$;

$$S_{26} = \frac{8(1-0.75^{26})}{1-0.75} = 31.98 \text{ to } 4 \text{ s.f.}$$

Example:



A plant grows 1.67 cm in its first week. Each week it grows by 4% more than it did the week before. By how much does it grow in nine weeks, including the first week?

Solution

The growths in the first 9 weeks are as follows :

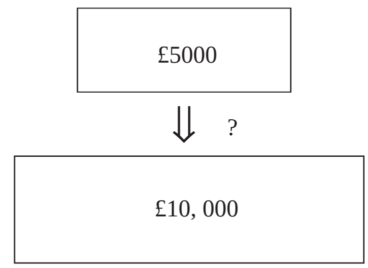
$$1.67,\ 1.67\times1.04,\ 1.67\times1.04^2, \ldots$$

Total growth in first nine weeks is

$$S_9 = \frac{1.67(1.04^9 - 1)}{1.04 - 1} = 17.67 \text{ cm to 4 s.f.}$$

Example:

After how many complete years will a starting capital of £5000 first exceed £10 000 if it grows at 6% per annum?



How many years later?

Solution

After n years, the capital sum has grown to

$$5000 \times (1.06)^n$$

When is this first greater than 10 000, n being a natural number?
In other words, the smallest value of n is required so that

$$5000 \times (1.06)^n > 10\,000, \; n \in \mathbb{N}$$

$$\Rightarrow \quad (1.06)^n > 2$$

Now take logs of both sides:

$$n \ln 1.06 > \ln 2$$

$$\Rightarrow \quad n > \frac{\ln 2}{\ln 1.06}$$

$$\Rightarrow \quad n > 11.9$$

After 12 years, the investment has doubled in value.

Activity 4 GP in disguise

(a) Why is this a geometric sequence?

1, –2, 4, –8, 16, ... ?

What is its common ratio? What is its nth term? What is S_n?

(b) Investigate in the same way, the sequence

1, –1, 1, –1, ...

Exercise 13A

1. Write down formulae for the nth term of these sequences:
   (a) 3, 6, 12, 24, ...
   (b) 36, 18, 9, 4.5, ...
   (c) 2, –6, 18, –54, ...
   (d) 90, –30, 10, $-3\frac{1}{3}$, ...
   (e) 10, 100, 1000, ...
   (f) 6, –6, 6, –6, ...
   (g) $\frac{1}{4}, \frac{1}{12}, \frac{1}{36}, \frac{1}{108}, \ldots$

2. Use the formula for S_n to calculate to 4 s.f.
   (a) 5 + 10 + 20 + ... to 6 terms
   (b) 4 + 12 + 36 + ... to 10 terms
   (c) $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+\ldots$ to 8 terms
   (d) 100 – 20 + 4 – ... to 20 terms
   (e) 16 +17.6 + 19.36 + ... to 50 terms
   (f) 26 – 16.25 + 10.15625 ... to 15 terms

3. Give the number (e.g. 12th term) of the earliest term for which
   (a) the sequence 1, 1.5, 2.25, ... exceeds 50;
   (b) the sequence 6, 8, $10\frac{2}{3}$, ... exceeds 250;
   (c) the sequence $\frac{2}{5}, \frac{1}{5}, \frac{1}{10}, \ldots$ goes below $\frac{1}{1000}$

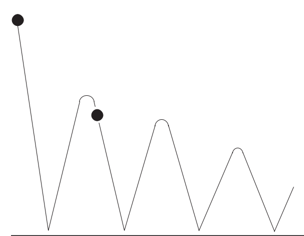
4. (a) For what value of n does the sum 50 + 60 + 72 + ... + $50 \times (1.2)^{n-1}$ first exceed 1000?
   (b) To how many terms can the following series be summed before it exceeds 2 000 000?

   2 + 2.01 + 2.02005 + ...

5. Dave invests £500 in a building society account at the start of each year. The interest rate in the account is 7.2% p.a. Immediately after he invests his 12th instalment he calculates how much money the account should contain. Show this calculation as the sum of a GP and use the formula for S_n to evaluate it.

13.2 Never ending sums

Many of the ideas used so far to illustrate geometric series have been to do with money. Here is one example that is not. If you drop a tennis ball, or any elastic object, onto a horizontal floor it will bounce back up part of the way. If left to its own devices it will continue to bounce, the height of the bounces decreasing each time.



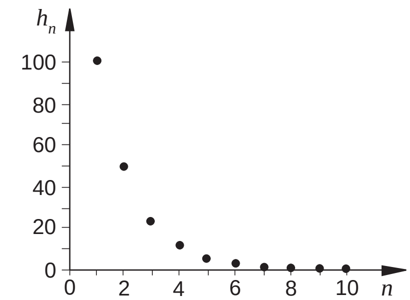
The ratio between the heights of consecutive bounces is constant, hence these heights follow a GP. The same thing is true of the times between bounces.

Activity 5 Bouncing ball

(a) A tennis ball is dropped from a height of 1 metre onto a concrete floor. After its first bounce, it rises to a height of 49cm. Call the height after the nth bounce h_n. Find a formula for h_n and say what happens to h_n as n gets larger.

(b) Under these circumstances the time between the first and second bounces is 0.6321 seconds. Call this t_1. The next time, t_2, is $0.7t_1$, and each successive time is 0.7 times the previous one. Find a formula for t_n.

(c) If $S_2 = t_1 + t_2$, what does S_2 represent? What does S_n mean? Calculate S_{10}, S_{20} and S_{50}. How long after the first bounce does the ball stop bouncing altogether, to the nearest tenth of a second?



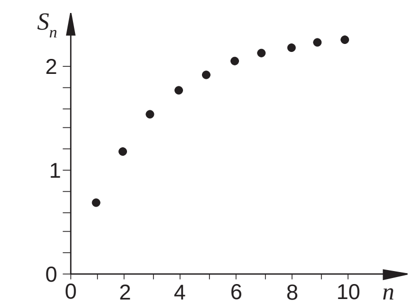
Activity 5 gave an example of a **convergent** sequence. Convergence, in this context, means that the further along the sequence you go, the closer you get to a specific value. For example, in part (a) the sequence to the nearest 0.1 cm is

100, 49.0, 24.0, 11.8, 5.8, 2.8, 1.4, ...

and the numbers get closer and closer to zero. Zero is said to be the **limit** of the sequence.

Part (b) also gave a sequence that converged to zero. In part (c), the sequence of numbers $S_1, S_2, S_3, \ldots$ start as follows :

0.6321, 1.0746, 1.3844, 1.6012, 1.7530, ...



You should have found that this sequence did approach a limit, but that this was not zero. Hence the series has a convergent **sum**, that is, the sum S_n of the series also converges.

The series 1, 2, 4, 8 ...... is a **divergent** sequence. It grows without limit as the number of terms increases. The same is true, in a slightly different sense, of the sequence 1, –2, 4, –8 ...... Any sequence that does **not** converge is said to be **divergent**.

Activity 6 Convergent or divergent?

For each of these sequences

(i) write a formula for the n^{th} term;

(ii) find whether the sequence converges;

(iii) find whether the sum S_n converges.

(a) $6, 2, \frac{2}{3}, \frac{2}{9}, \ldots$

(b) 1, 1.5, 2.25, 3.375, ...

(c) $4, -3, \frac{9}{4}, -\frac{27}{16} \ldots$

(d) 1, 1.01, 1.012, 1.0123, ...

(e) 8, –9.6, 11.52, –13.824 ...

Activity 7 Behaviour of r^n

(a) What happens to r^n as n gets larger (i.e. as $n \to \infty$)? (You might need to see what happens for a variety of different values of r, positive and negative, large and small.)

(b) The sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

If r^n converges to 0 as $n \to \infty$, what can you say about the limit of S_n as $n \to \infty$?

Activity 8 Experimental verification

Conduct an experiment with a bouncing ball. Calculate the theoretical time from the first bounce until it stops bouncing. Then use a stop-watch to see how close the answer is to your calculation. You will need to know that:

If a ball is dropped from a height of 1 metre and rises after the first bounce to a height of h metres, then the time between the first and second bounce is given by

$$t_1 = 0.90305\sqrt{h},$$

and the common ratio in the sequence $t_1, t_2, t_3 \ldots$ is $\sqrt{h}$.

A geometric series, $a + ar + ar^2 + \ldots + ar^{n-1}$ converges when $|r| < 1$; i.e. for $-1 < r < 1$. Since if $|r| < 1$, $r^n \to 0$ as $n \to \infty$ and

$$\boxed{S_n \to \frac{a}{1-r} \text{ as } n \to \infty}$$

The limit $\dfrac{a}{1-r}$ is known as the 'sum to infinity' and is denoted S_∞.

Example

Find

(a) 8 + 4 + 2 + 1+ ...

(b) 20 – 16 + 12.8 – 10.24 + ...

Solution

(a) This is a geometric series with first term 8 and common ratio $\frac{1}{2}$, so

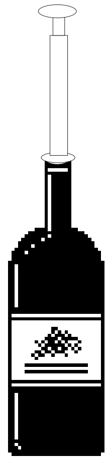
$$S_\infty = \frac{8}{1-\frac{1}{2}} = 16.$$

(b) This is a geometric series with first term 20 and common ratio – 0.8, so

$$S_\infty = \frac{20}{1-(-0.8)} = \frac{20}{1.8} = \frac{100}{9} = 11.1 \text{ (to 3 s.f.).}$$

Exercise 13B

1. Find these sums to infinity, where they exist.
   (a) 80 + 20 + 5 + 1.25 + ...
   (b) 180 – 60 + 20 – 20/3 + ...
   (c) 2 + 1.98 + 1.9602 + ...
   (d) –100 + 110 – 121 + ...
   (e) 1/10 + 1/100 + 1/1000 + ...
2. (a) What is 1/10 + 1/100 + 1/1000 + ... as a recurring decimal?
   (b) Express 0.37373737 ... as an infinite geometric series and find the fraction it represents.
3. What fractions do these decimals represent?
   (a) 0.52525252 ...
   (b) 0.358358358...
   (c) 0.194949494 ...
4. (a) A GP has a common ratio of 0.65. Its sum to infinity is 120. What is the first term?
   (b) Another GP has 2.8 as its first term and its sum to infinity is 3.2 Find its common ratio.
5. Rosita is using a device to extract air from a bottle of wine. This helps to preserve the wine left in the bottle.

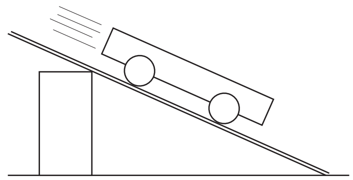
   The pump she uses can extract a maximum of 46 cm^3. In practice what happens is that the first attempt extracts 46 cm^3 and subsequent extractions follow a geometric sequence.

   Rosita's second attempt extracts 36 cm^3. What is the maximum amount of air she can remove in total?
6. A rubber ball is dropped from a height of 6 metres and after the first bounce rises to a height of 4.7 m. It is left to continue bouncing until it stops.
   (a) A computerized timer is started when it first hits the ground. The second contact with the ground occurs after 1.958 seconds and the third after 3.690 seconds. Given that the times between consecutive contacts with the ground follow a geometric sequence, how long does the ball take to stop bouncing?
   (b) The heights to which the ball rises after each impact also follow a geometric sequence. Between the release of the ball and the second bounce the ball travels 6 + 2 x 4.7 = 15.4 m. How far does the ball travel altogether?

13.3 Arithmetic sequences

Geometric sequences involve a constant ratio between consecutive terms. Another important type of sequence involves a constant **difference** between consecutive terms; such a sequence is called an **arithmetic** sequence.

In an experiment to measure the descent of a trolley rolling down a slope a 'tickertape timer' is used to measure the distance travelled in each second. The results are shown in the table.



Second	cm travelled in second
1	3
2	5
3	7
4	9
5	11
6	13

The sequence 3, 5, 7, 9, 11, 13 is an example of an arithmetic sequence. The sequence starts with 3 and thereafter each term is 2 more than the previous one. The difference of 2 is known as the **common difference**.

It would be useful to find the total distance travelled in the first 6 seconds by adding the numbers together. A quick numerical trick for doing this is to imagine writing the numbers out twice, once forwards once backwards, as shown below

3	5	7	9	11	13
13	11	9	7	5	3

Each pair of vertical numbers adds up to 16. So adding the two sequences, you have 6×16 between them. Hence the sum of the original series is

$$\frac{1}{2} \times (6 \times 16) = 48.$$

The sum of terms of an arithmetic sequence is called an **arithmetic series** or **progression**, often called AP for short.

Activity 9 Distance travelled

Use the example above of a trolley rolling down a slope to answer these questions.

(a) Work out the distance travelled in the 20th second.

(b) Calculate S_{20}, the distance travelled in the first 20 seconds, using the above method.

(c) What is the distance travelled in the nth second?

(d) Show that the trolley travels a distance of $n(n+2)$ cm in the first n seconds.

Example

Consider the arithmetic sequence 8, 12, 16, 20 ...

Find expressions

(a) for u_n, (the nth term) (b) for S_n.

Solution

In this AP the first term is 8 and the common difference 4.

(a) $u_1 = 8$

$u_2 = 8 + 4$

$u_3 = 8 + 2 \times 4$

$u_4 = 8 + 3 \times 4$ etc.

u_n is obtained by adding on the common difference $(n-1)$ times.

$$\Rightarrow \quad u_n = 8 + 4(n-1)$$

$$= 4n + 4$$

(b) To find S_n, follow the procedure explained previously:

$8 \quad 12 \quad \ldots\ldots\ldots\ldots \quad 4n \quad 4n+4$

$4n+4 \quad 4n \quad \ldots\ldots\ldots\ldots \quad 12 \quad 8$

Each pair adds up to $4n + 12$. There are n pairs.

So $\quad 2S_n = n(4n+12)$

$= 4n(n+3)$

giving $\quad S_n = 2n(n+3)$.

Exercise 13C

1. Use the 'numerical trick' to calculate
   (a) 3 + 7 + 11 + ... + 27
   (b) 52 + 46 + 40 + ... + 4
   (c) the sum of all the numbers on a traditional clock face;
   (d) the sum of all the odd numbers between 1 and 99.

2. Find formulae for u_n and S_n in these sequences :
   (a) 1, 4, 7, 10, ...
   (b) 12, 21, 30, 39, ...
   (c) 60, 55, 50, 45, ...
   (d) 1, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, ...

3. A model railway manufacturer makes pieces of track of lengths 8 cm, 10 cm, 12 cm, etc. up to and including 38 cm. An enthusiast buys 5 pieces of each length. What total length of track can be made?

The general arithmetic sequence is often denoted by

a, $a+d$, $a+2d$, $a+3d$, etc. ...

To sum the series of the first n terms of the sequence,

$$S_n = a + (a+d) + (a+2d) + \ \dots \ + (a + (n-1)d)$$

Note that the order can be reversed to give

$$S_n = (a + (n-1)d) + (a + (n-2)d) + \ \dots \ + a$$

Adding the two expressions for S_n gives

$$\begin{aligned} 2S_n &= (2a + (n-1)d) + (2a + (n-1)d) + \ \dots \ + (2a + (n-1)d) \\ &= n[2a + (n-1)d] \end{aligned}$$

So

$$\boxed{S_n = \frac{n}{2}(2a + (n-1)d)}$$

An alternative form for S_n is given in terms of its first and last term, a and l, where

$$l = a + (n-1)d$$

since the nth term of the sequence is given by

$$u_n = a + (n-1)d.$$

Thus

$$S_n = \frac{n}{2}(a + l)$$

Example

Sum the series 5 + 9 + 13 + ... to 20 terms.

Solution

This is an arithmetic sequence with first term 5 and common difference 4; so

$$S_{20} = \frac{20}{2}(2 \times 5 + 19 \times 4) = 860$$

Example

The sum of the series 1 + 8 + 15 + ... is 396. How many terms does the series contain?

Solution

This is an arithmetic sequence with first term 1 and common difference 7. Let the number of terms in the sequence be n.

$$S_n = 396$$

$$\Rightarrow \quad \frac{n}{2}\left(2+7(n-1)\right) = 396$$

$$\Rightarrow \quad n(7n-5) = 792$$

$$\Rightarrow \quad 7n^2 - 5n - 792 = 0$$

$$\Rightarrow \quad (7n+72)(n-11) = 0$$

$$\Rightarrow \quad n = 11 \text{ since } -\frac{72}{7} \text{ is not an integer.}$$

The number of terms is 11.

Activity 10 Ancient Babylonian problem

Ten brothers receive 100 shekels between them. Each brother receives a constant amount more than the next oldest. The seventh oldest brother receives 7 shekels. How much does each brother receive?

Exercise 13D

1. Find the sum of
   (a) 11 + 14 + 17 + ... to 16 terms
   (b) 27 + 22 + 17 + ... to 10 terms
   (c) 5 + 17 + 29 + ... + 161
   (d) 7.2 + 7.8 + 8.4 + ... to 21 terms
   (e) 90 + 79 + 68 + ... –20
   (f) 0.12 + 0.155 + 0.19 + ... to 150 terms
2. The last three terms of an arithmetical sequence with 18 terms is as follows : ...67, 72, 77. Find the first term and the sum of the series.
3. How many terms are there if
   (a) 52 + 49 + 46 + ... = 385?
   (b) 0.35 + 0.52 + 0.69 + ... = 35.72?
4. The first term of an arithmetic series is 16 and the last is 60. The sum of the arithmetic series is 342. Find the common difference.
5. New employees joining a firm in the clerical grade receive an annual salary of £8500. Every year they stay with the firm they have a salary increase of £800, up to a maximum of £13300 p.a. How much does a new employee earn in total, up to and including the year on maximum salary?

13.4 Sigma notation

Repeatedly having to write out terms in a series is time consuming. Mathematicians have developed a form of notation which both shortens the process and is easy to use. It involves the use of the Greek capital letter Σ (sigma), the equivalent of the letter S, for sum.

The series $2 + 4 + 8 + \ldots + 2^{12}$ can be shortened to $\sum_{r=1}^{12} 2^r$.

This is because every term in the series is of the form 2^r, and all the values of 2^r, from $r=1$ to $r=12$ are added up. In this example the '2^r' is called the **general term**; 12 and 1 are the top and bottom **limits** of the sum.

Similarly, the series

$$60 + 60\times(0.95) + \ldots + 60\times(0.95)^{30}$$

can be abbreviated to

$$\sum_{r=0}^{30} 60\times(0.95)^r.$$

Often there is more than one way to use the notation. The series

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ \ldots\ +\frac{99}{100}$$

has a general term that could be thought of as either

$\dfrac{r}{r+1}$ or as $\dfrac{r-1}{r}$. Hence the series can be written as either

$$\sum_{r=1}^{99}\frac{r}{r+1} \text{ or } \sum_{r=2}^{100}\frac{r-1}{r}.$$

Example

Write out what $\sum_{r=1}^{9}(10-r)^2$ means and write down another way of expressing the same series, using Σ notation.

Solution

$$\sum_{r=1}^{9}(10-r)^2 = (10-1)^2 + (10-2)^2 + \ldots + (10-9)^2$$

$$= 9^2 + 8^2 + \ldots + 1^2$$

An alternative way of writing the same series is to think of it in reverse:

$$1^2 + 2^2 + \ldots + 8^2 + 9^2 = \sum_{r=1}^{9} r^2$$

Example

Express in Σ notation 'the sum of all multiples of 5 between 1 and 100 inclusive'.

Solution

All multiples of 5 are of the form $5r, r \in \mathbb{N}$.

$100 = 5 \times 20$, so the top limit is 20. The lowest multiple of 5 to be included is 5×1. The sum is therefore

$$5 + 10 + 15 + \ldots + 100 = \sum_{r=1}^{20} 5r$$

Example

Express in Σ notation 'the sum of the first n positive integers ending in 3'.

Solution

Numbers ending in 3 have the form $10r+3$, $r \in \mathbb{N}$. The first number required is 3 itself, so the bottom limit must be $r = 0$. This means that the top limit must be $n-1$. Hence the answer is

$$\sum_{r=0}^{n-1}(10r+3) \qquad (= 3 + 13 + \ldots + (10n-7))$$

[An alternative answer is $\sum_{r=1}^{n}(10r-7)$]

Exercise 13E

1. Write out the first three and last terms of:

   (a) $\sum_{r=5}^{15} r^2$ (b) $\sum_{r=1}^{10}(2r-1)$

   (c) $\sum_{r=1}^{n} r$ (d) $\sum_{r=3}^{10}\frac{r-2}{r}$

   (e) $\sum_{r=6}^{100}(r-2)^2$

2. Shorten these expressions using Σ notation.

   (a) $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{25}$

   (b) $10+11+12+\ldots+50$

   (c) $1+8+27+\ldots+n^3$

   (d) $1+3+9+27+\ldots+3^{12}$

   (e) $6+11+16+\ldots+(5n+1)$

   (f) $14+17+20+\ldots+62$

   (g) $5+50+500+\ldots+5\times10^n$.

   (h) $\frac{1}{6}+\frac{2}{12}+\frac{3}{20}+\ldots+\frac{20}{21\times22}$

3. Use Σ notation to write:

   (a) the sum of all natural numbers with two digits;

   (b) the sum of the first 60 odd numbers;

   (c) the sum of all the square numbers from 100 to 400 inclusive;

   (d) the sum of all numbers between 1 and 100 inclusive that leave remainder 1 when divided by 7.

4. Find alternative ways, using Σ notation, of writing these:

   (a) $\sum_{r=1}^{19}(20-r)$ (b) $\sum_{r=2}^{41}\frac{1}{r-1}$ (c) $\sum_{r=-3}^{3} r^2$

13.5 More series

Activity 11

(a) Write down the values of $(-1)^0$, $(-1)^1$, $(-1)^2$, $(-1)^3$ etc. Generalise your answers.

(b) Write down the first three terms and the last term of

(i) $\sum_{r=0}^{10}(-1)^r\frac{1}{2^r}$ (ii) $\sum_{r=0}^{10}(-1)^{r+1}\left(\frac{r^2}{r^2+1}\right)$

(c) How can you write the series

$100-100\times(0.8)+100\times(0.8)^2 \ \ldots$ to n terms

using Σ notation?

Activity 12 Properties of Σ

(a) Calculate the numerical values of

$$\sum_{r=1}^{5} r \qquad \sum_{r=1}^{5} r^2 \qquad \sum_{r=1}^{5} r^3 \qquad \sum_{r=1}^{5}\left(r+r^2\right) \qquad \sum_{r=1}^{5} 3r$$

(b) If $u_1, u_2, \ldots u_n$ and $v_1, v_2, \ldots, v_n$ are two sequences of numbers, is it true that

$$\sum_{r=1}^{n}(u_r + v_r) = \sum_{r=1}^{n} u_r + \sum_{r=1}^{n} v_r \,?$$

Justify your answer.

(c) Investigate the truth or falsehood of these statements:

(i) $\sum_{r=1}^{n} u_r v_r = \left(\sum_{r=1}^{n} u_r\right)\left(\sum_{r=1}^{n} v_r\right)$

(ii) $\sum_{r=1}^{n} u_r^2 = \left(\sum_{r=1}^{n} u_r\right)^2$

(iii) $\sum_{r=1}^{n} \alpha u_r = \alpha\left(\sum_{r=1}^{n} u_r\right)$ [α is any number.]

Again, justify your answers fully.

(d) What is the value of $\sum_{r=1}^{5}(r+1)$?

What is $\sum_{r=1}^{n} 1$? and $\sum_{r=1}^{n} r$?

Exercise 13F

1. Work out the numerical value of

(a) $\sum_{r=1}^{10} 1$ (b) $\sum_{r=1}^{25} 4$

(c) $\sum_{r=0}^{16} (3+5r)$ (d) $\sum_{r=0}^{30} 3\times(3.5)^r$

(e) $\sum_{r=1}^{\infty} (0.7)^r$ (f) $\sum_{r=0}^{\infty} 5\times(-\frac{2}{3})^r$

2. Use $\sum$ notation to write these:

(a) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ \ldots\ -\frac{1}{6}$

(b) $-1+4-9+16-\ \ldots\ +144$

(c) $12-12\times0.2+12\times0.04-\ \ldots\ +12\times(0.2)^{50}$

3. If you know that

$\sum_{r=1}^{n} u_r = 20$ and $\sum_{r=1}^{n} v_r = 64$

calculate where possible:

(a) $\sum_{r=1}^{n}(u_r+v_r)$ (b) $\sum_{r=1}^{n} u_r v_r$

(c) $\sum_{r=1}^{n} u_r^2$ (d) $\sum_{r=1}^{n} \frac{1}{2} v_r$

(e) $\sum_{r=1}^{n}(v_r-u_r)$ (f) $\sum_{r=1}^{n}(5u_r-v_r)$

(g) $\sum_{r=1}^{2n} u_r$ (h) $\sum_{r=1}^{n}(-1)^r v_r$

*13.6 Useful formulae

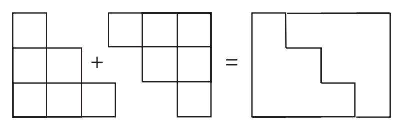
You will find it useful to know three important results

- a formula for $\sum_{r=1}^{n} r \quad (1+2+3+\ \dots\ +n)$
- a formula for $\sum_{r=1}^{n} r^2 \quad \left(1^2+2^2+3^2+\ \dots\ +n^2\right)$
- a formula for $\sum_{r=1}^{n} r^3 \quad \left(1^3+2^3+3^3+\ \dots\ +n^3\right)$

The following few activities are designed to illustrate cases where these series arise, and to work out what these formulae are.

Activity 13 Sum of first n natural numbers

Find a formula for $\sum_{r=1}^{n} r$ by



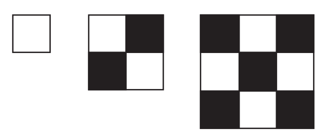
(a) treating the sum as an arithmetic series:

(b) using a geometrical argument based on the diagram on the right;

(c) starting from the statement

$$\sum_{r=1}^{n} r = \sum_{r=1}^{n}\left((n+1)-r\right).$$

Activity 14 Another chess board problem



(a) How many squares are there in each of the 'mini chessboards' on the right?

(b) How many squares are there on a chessboard? Express your answer using $\sum$ notation.

(c) What is a corresponding three dimensional problem?

Activity 15 Sum of squares

(a) Work out $\sum_{r=1}^{n} r(r+1)$ for $n = 1, 2, 3, \ldots 8$.

Copy and complete the table of results opposite.

(b) Prepare formula for $\sum_{r=1}^{n} r(r+1)$.

(c) Use your formulae for $\sum_{r=1}^{n} r(r+1)$ and $\sum_{r=1}^{n} r$ to obtain a formula for $\sum_{r=1}^{n} r^2$.

n	$\sum_{r=1}^{n} r(r+1)$
1	2
2	8
3	20
4	...
5	...
6	...
7	168
8	...

Activity 16 Sum of cubes

(a) For n = 1, 2, 3 and 4, work out $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^3$.

Conjecture a formula for $\sum_{r=1}^{n} r^3$.

(b) Prove this formula by starting from the statement

$$\sum_{r=1}^{n} r^3 = \sum_{r=1}^{n} \left((n+1)-r\right)^3 .$$

Why could a similar approach not be used to prove the formula for $\sum r^2$?

The results of the last few activities can be summarised as follows.

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$

The useful fact that $\sum r^3 = \left(\sum r\right)^2$ is a coincidence (if there is such a thing in maths). It is not possible to extend this to find $\sum r^4, \sum r^5$ etc. Formulae do exist for sums of higher powers, but they are somewhat cumbersome and seldom useful.

Exercise 13G

1. Write down the general term, and hence evaluate:
   (a) $1+2+3+ \ldots +20$
   (b) $1^2+2^2+3^2+ \ldots +10^2$
   (c) $2+8+18+ \ldots +(2\times 15^2)$
   (d) $2+4+6+ \ldots +100$
   (e) $1+3+5+ \ldots +25$
   (f) $1+8+27+ \ldots +1000$

2. Work out $\sum_{r=10}^{20} r$. Use the fact that

$$\sum_{r=10}^{20} r = \sum_{r=0}^{20} r - \sum_{r=0}^{9} r.$$

3. Use techniques similar to that in Question 2 to calculate
   (a) $11^2+12^2+ \ldots +24^2$
   (b) $7^3+8^3+\ldots +15^3$
   (c) $21+23+ \ldots +61$

4. Calculate 21+23+25 ... +161

5. Prove that $\sum_{r=0}^{2n} r = n(2n+1)$

   and hence that $\sum_{r=n+1}^{2n} r = \frac{1}{2}n(3n+1)$

13.7 Miscellaneous Exercises

1. A piece of paper is 0.1 mm thick. Imagine it can be folded as many times as desired. After one fold, for example, the paper is 0.2 mm thick, and so on.
   (a) How thick is the folded paper after 10 folds?
   (b) How many times should the paper be folded for its thickness to be 3 metres or more? (About the height of a room.)
   (c) How many more times would it need to be folded for the thickness to reach the moon? (400 000 km away)

2. Find formulae for the *n*th terms of each of these sequences:
   (a) 4, 6, 9, 13.5, ...
   (b) 250, 244, 238, 232, ...
   (c) 10, 2, 0.4, 0.08, ...
   (d) 0.17, 0.49, 0.81, 1.13, ...

3. Evaluate these sums:
   (a) 12+15+18+21+ ... to 20 terms;
   (b) 4+12+36+108+ ... to 12 terms;
   (c) 5–2–9–16– ... –65;
   (d) 240+180+135+ ... to infinity.

4. The first term of a geometric sequence is 7 and the third term is 63. Find the second term.

5. Consider the arithmetic series 5+9+13+17+...
   (a) How many terms of this series are less than 1000?
   (b) What is the least value of *n* for which $S_n > 1000$?

6. The ninth term of an arithmetic progression is 52 and the sum of the first twelve terms is 414. Find the first term and the common difference. (AEB)

7. In an arithmetic progression, the eighth term is twice the 3rd term and the 20th term is 110.
   (a) Find the common difference.
   (b) Determine the sum of the first 100 terms.
   (AEB)

8. The first term of a geometric progression is 8 and the sum to infinity is 400.
   (a) Find the common ratio.
   (b) Determine the least number of terms with a sum greater than 399. (AEB)

9. The sum of the first and second terms of a geometric progression is 108 and the sum of the third and fourth term is 12. Find the two posible values of the common ratio and the corresponding values of the first term. (AEB)