12 INTEGRATION

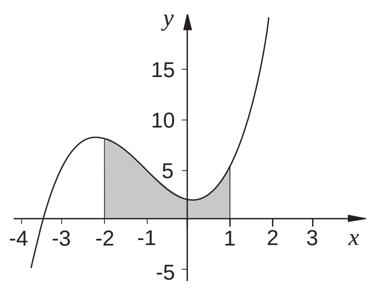
Objectives

After studying this chapter you should

- understand the concept of integration;
- appreciate why finding the area under a graph is often important;
- be able to calculate the area under a variety of graphs given their equation.

12.0 Introduction

Integration is the process of finding the area under a graph. An example of an area that integration can be used to calculate is the shaded one shown in the diagram. There are several ways of **estimating** the area - this chapter includes a brief look at such methods - but the main objective is to discover a way to find the area **exactly**.



$y = x^3 + 3x^2 - x + 2$

Finding the area under a graph is not just important for its own sake. There are a number of problems in science and elsewhere that need integration for a solution. Section 12.1 starts by looking at an example of such a problem.

12.1 Estimating future populations

Shortly after the Second World War, it was decided to establish several 'new towns'. Two well known examples are Hemel Hempstead, near London, and Newton Aycliffe in the north-east; in these cases the 'new towns' were based on existing small communities. More recently, the city of Milton Keynes was built up where once there was practically nothing.

Clearly such ambitious developments require careful planning and one factor that needs to be considered is population growth. Apart from anything else, the services and infrastructure of a new community need building up in accordance with the projected population: chaos would ensue, for instance, if there were 5000 children but only enough schools for 3000.

Imagine you are planning a new town. You have been advised that planned population growth will conform to the following model:

Initial growth rate 6000 people per year;
thereafter the rate of growth will decrease
by 30% every five years.

Your task is to estimate what the total population will be 30 years later. One approach to this problem is detailed in the following activity.

Activity 1 Rough estimates

(a) Start by setting up a mathematical model. Let t stand for the time in years; when $t = 0$ the growth rate is 6000 people per year. When $t = 5$ (after 5 years) the growth rate is 6000 less 30%, i.e. $0.7 \times 6000 = 4200$.

Copy and complete this table of growth rates :

Time (years)	Growth Rate
0	6000
5	4200
10	2940
15	...
20	...
25	...
30	706

(b) A very rough estimate can be obtained by following this line of reasoning:

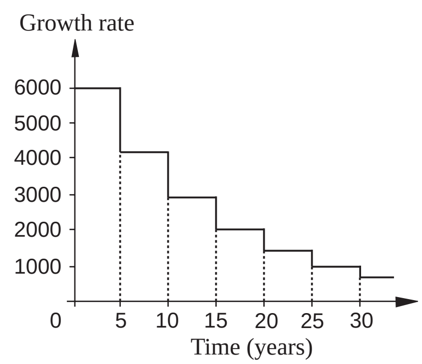
Suppose the growth rate remains fixed at 6000 throughout the first 5 years. Then after 5 years the population will be

$$5 \times 6000 = 30\ 000.$$

Now suppose the growth rate over the next five years is a constant 4200 per year. After 10 years the population will be

$$30\ 000 + 5 \times 4200 = 51\ 000.$$

Continue this process to obtain an estimate for the population after 30 years.



(c) Your figure for (b) will clearly be an over-estimate. A similar process can be applied to give an under-estimate. The process starts as follows:

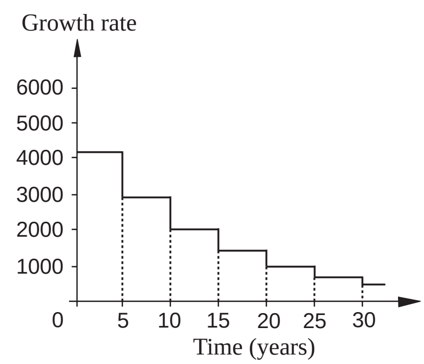
Suppose the growth rate over the first five years is fixed at 4200. Then after five years the population will be

$$5 \times 4200 = 21\ 000$$

After 10 years the population will be

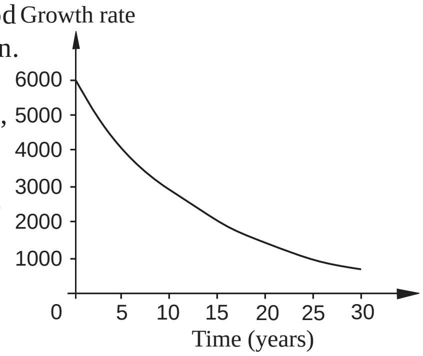
$$21\ 000 + 5 \times 2940 = 35\ 700.$$

Continue this process to obtain a second estimate.

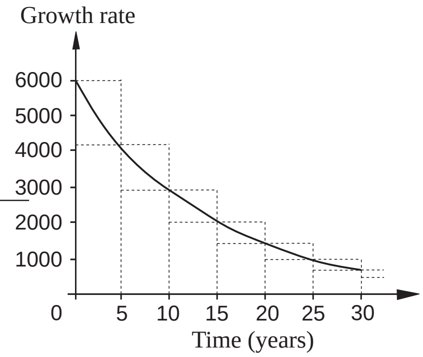


(d) Use your two estimates to make a third, more realistic estimate of the population after 30 years' growth.

The two diagrams alongside the above activity should give a good idea of what the processes you have just completed actually mean. In parts (b) and (c) what you essentially did was to find the area under a bar chart; one estimate was too large, the other too small, with the 'true' answer lying somewhere in the middle.



The graph of growth rate against time will not, in truth, be a 'bar chart' at all. More realistically it will resemble the curve shown opposite, which shows the growth rate decreasing continuously. The figure below shows the same curve with the two bar charts superimposed.



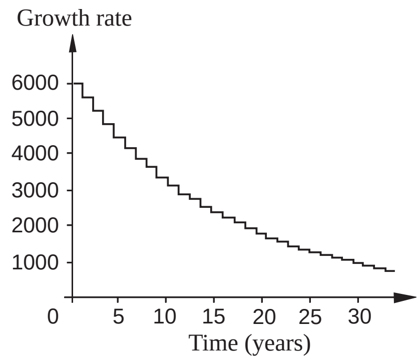
The 'true' population estimate will be the **exact** area under the curve in the figure above.

Activity 2 More accurate estimate

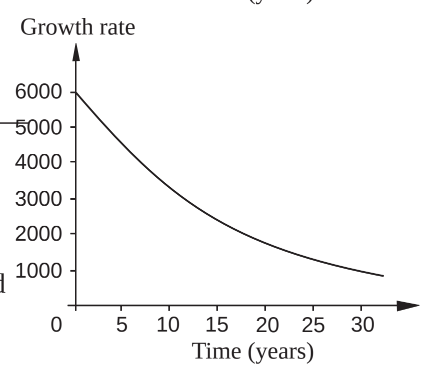
A more refined estimate can be obtained by using growth rates calculated at yearly intervals rather than 5-yearly ones. A decrease of 30% every **five** years is roughly equivalent to a decrease of 6.9% every year.

Hence the table of growth rates starts like this:

Time (years)	Growth rate
0	6000
1	5586
2	5201
etc.	etc.



Use these figures to get a closer estimate of the population. Follow the same sort of process as in Activity 1.



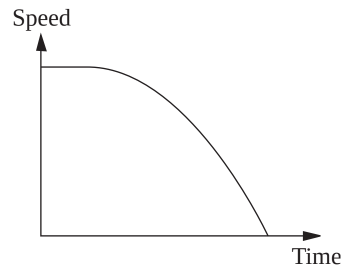
Why was the figure 6.9% used in this activity?

With a little ingenuity you might be able to save tedious calculation by efficient use of a computer. If you do this, try and refine the estimate still further by using smaller intervals.

The graphs alongside Activity 2 illustrate why using smaller intervals leads to increased accuracy. The smaller the intervals, the more the 'bar chart' resembles the continuous curve. Hence the area of the 'bar charts' approaches the exact value of the area under the curve as the interval size is decreased.

The next activity provides a further example of finding estimates for the area under a curve.

Activity 3 Distance travelled



(a) A car is travelling at 36 metres per second when the driver spots an obstruction ahead. The car does an 'emergency stop'; the speed of the car from the moment the obstruction is spotted is shown in the table below.

Time (s)	0	1	2	3	4	5	6	7
Speed (ms^{-1})	36	36	34.8	29.9	23.2	15.2	4.8	0

As accurately as you can, draw a speed-time curve describing the car's motion. (The graph opposite is not accurate but shows the general shape).

(b) How far did the car travel in the first second?

(c) If the car travelled at 36 ms^{-1} **throughout** the 2nd second, how far would it have travelled in this second?

(d) If the car travelled at 34.8 ms^{-1} throughout the 3rd second, how far would it have travelled in this second?

(e) Following this procedure up to the 7th second inclusive; work out an estimate of the distance travelled by the car from the moment the obstruction was spotted.

(f) The answer to (e) is an over-estimate. Use the ideas of Activity 1 to produce an under-estimate.

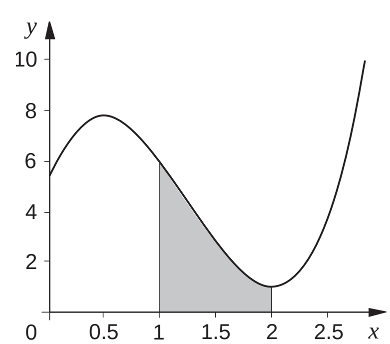
(g) Superimpose two bar charts onto your graph, the areas under which are your answers to (e) and (f).

(h) Use your answers so far to write down a better estimate of the distance travelled.

(i) Increased accuracy can be obtained by halving the interval width. First use your graph to estimate the car's speed after 0.5 seconds, 1.5 seconds, etc. Record your answers clearly.

(j) Now use these figures to produce a more accurate estimate than your answer to (h).

Exercise 12A



1. The sketch opposite shows the graph of the curve with equation

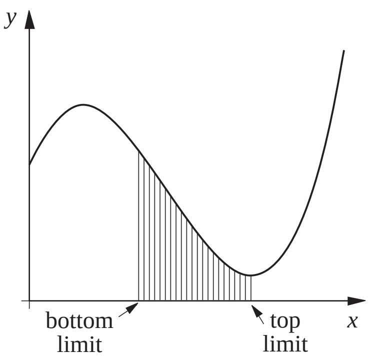
$$y = 4x^3 - 15x^2 + 12x + 5.$$

(a) Copy and complete this table of values :

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	6	5.37	..	..	..	..	2.18	..	..	..	1

(b) Use your table to estimate the shaded area in the diagram. Employ methods like those in the earlier questions and activities.

12.2 Notation for area



Three examples have now been encountered in which the area under a graph has been of significance. Mathematicians have for many centuries appreciated the importance of areas under curves. The process of working out such areas is called 'integration' and early approaches to the problem were similar to the method investigated so far, namely the splitting of areas into several thin rectangles.

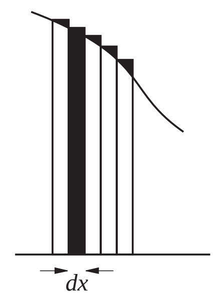
The notation developed by the German mathematician, *Leibnitz*, quickly became widely adopted and is still used today. The area under the curve

$$y = 4x^3 - 15x^2 + 12x + 5$$

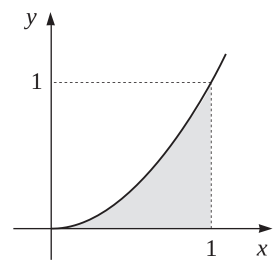
between $x = 1$ and $x = 2$ is written as

$$\int_1^2 \left(4x^3 - 15x^2 + 12x + 5\right) dx,$$

and is read as the integral between 1 and 2 of $4x^3 - 15x^2 + 12x + 5$.



The $\int$ sign, known as the **integral sign**, derives from the ancient form of the letter S, for sum. The 'dx' represents the width of the small rectangles. The above notation denotes the sum of lots of very thin rectangles between the limits $x = 1$ and $x = 2$.



Hence the area under the curve $y = x^2$ from $x = 0$ to $x = 1$ is written as

$$\int_0^1 x^2 dx$$

The $\int$ and the 'dx' enclose the function to be integrated.

Area under straight line

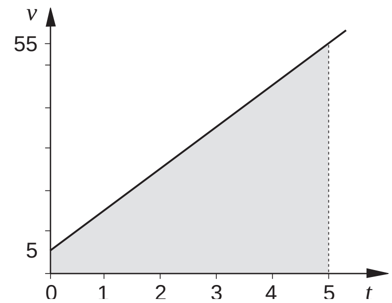
The activities in the previous section highlighted the fact that, at present, the methods used to calculate areas have only been approximate. When the graph is a straight line, however, it is a simple matter to work out the area exactly.

As you have seen, the area under a velocity-time graph gives the distance travelled. In many situations, particularly when a body is moving freely under gravity, the velocity-time graph is a straight line.

Consider the case of a stone thrown vertically downwards from a cliff-top. If its initial velocity is 5 metres per second then its velocity v metres per second after t seconds will be given approximately by the formula

$$v = 5 + 10t.$$

How far will it have travelled after 5 seconds, assuming it hasn't hit anything by then? The answer to this question can be worked out by finding the area under, that is integrating, the velocity-time graph.



The problem thus boils down to finding the shaded area in the diagram opposite. Using the integral sign, this area can be written

$$\int_0^5 (5 + 10t)\, dt.$$

(Note that *dt* is used here rather than *dx* as the variable along the horizontal axis is *t*).

The shaded area is a trapezium, the area of which can be worked out using the general rule

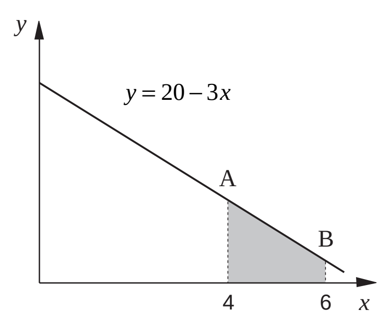
$$\text{area} = \frac{(\text{sum of parallel sides}) \times (\text{distance between them})}{2}$$

In this case the calculation yields $\dfrac{(55+5)\times 5}{2} = 150.$

So in 5 seconds the stone will have fallen 150 metres.

Example

Calculate the integral $\int_4^6 (20 - 3x)\, dx$.



Solution

The area required is shown in this sketch. Again, it is a trapezium.

To work out the lengths of the parallel sides you need to know the y-coordinates of A and B. These can be worked out from the equation of the line, $y = 20 - 3x$.

At A: $y = 20 - (3 \times 4) = 8.$

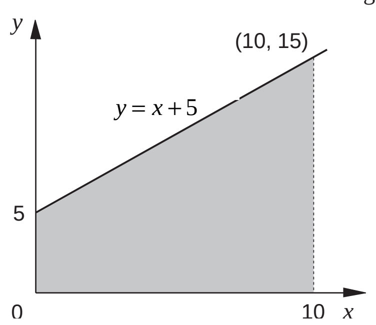
At B: $y = 20 - (3 \times 6) = 2.$

So the area is $\frac{(8+2)\times 2}{2} =$ 10 square units.

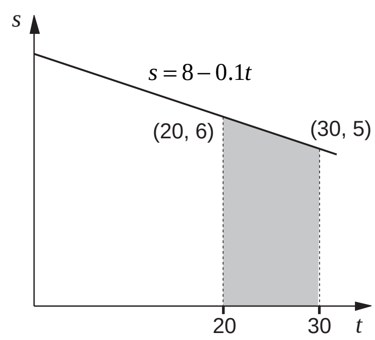
Exercise 12B

1. Calculate the shaded areas in these diagrams.

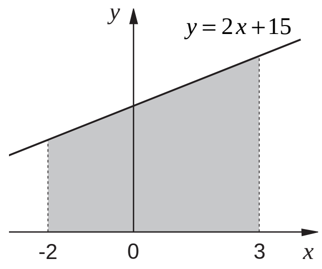
(a)



(b)



(c)



2. Express the areas in Question 1 using the integral sign. (Warning : don't automatically use 'dx'.)

3. Two quantities p and q are related by the equation

$$q = 12p - 45.$$

(a) Sketch the graph of this relationship for values of p between 0 and 10.

(b) Calculate

$$\int_5^8 (12p - 45)\ dp.$$

and shade on your graph the area this represents.

4. Calculate these integrals. Draw a sketch diagram if it helps.

(a) $\int_1^6 (6x+3)\,dx$

(b) $\int_{-1}^1 (25-7t)\,dt$

(c) $\int_{20}^{75} (-p+100)\,dp$

(d) $\int_{2.7}^{6.5} (3.5s+17.1)\,ds$

5. The speed of a car as it rolls up a hill is given by $v = 20 - 3t$ where t is the time in seconds and v is measured in metres per second.

(a) Draw a sketch graph showing speed against time between $t = 0$ and $t = 5$.

(b) Integrate to find how far the car travels in the first 5 seconds.

6. The rate at which a city's population grows is given by the formula

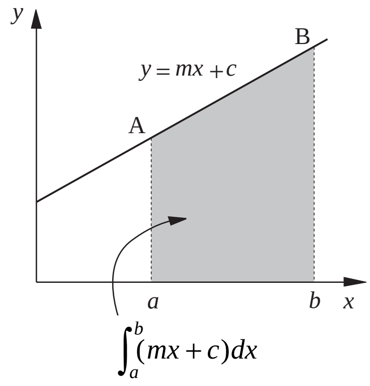
$$R = 1000 + 700t$$

where R is the rate of increase in people per year and t is the number of years since 1st January 1990.

(a) How fast will the population be growing on 1st January 1993, according to this model?

(b) The population starts from zero on 1st January 1990. Calculate, by integration, what the population should be on 1st January 1993. (Draw a sketch graph.)

12.3 General formula



It is straight forward to find the area under any straight line graph, say

$$\boxed{y = mx + c}$$

between $x = a$ and $x = b$, as shown in the figure opposite.

What are the y coordinates of A and B?

The area of the trapezium is given by

$$\tfrac{1}{2}((mb+c)+(ma+c))(b-a)$$

$$= \tfrac{1}{2}(mb+ma+2c)(b-a)$$

$$= \tfrac{1}{2}(mb^2+mab+2cb-mab-ma^2-2ca)$$

$$\boxed{\text{area } = \left(\tfrac{1}{2}mb^2+cb\right)-\left(\tfrac{1}{2}ma^2+ca\right)}$$

You can use this formula to check your answer to Question 1 in Exercise 12B.

The formula is usually written $\left[\tfrac{1}{2}mx^2+cx\right]_a^b$

This is short for the 'function $\tfrac{1}{2}mx^2+cx$ evaluated at the top limit, $x = b$, minus the value of the same function at the lower limit, $x = a$'.

The function $\tfrac{1}{2}mx^2+cx$ can thus be used to find areas efficiently. It is called an **indefinite integral** of the function $mx+c$, since in this form it has no limits. One way of thinking of it is as the ‘area function’ for the graph of $y = mx + c$. For the moment the ‘area function’, or indefinite integral, will be denoted by $A(x)$, whereas the area between $x = a$ and $x = b$ is given by

$$\text{Area } = \int_a^b (mx+c)dx = \left[\tfrac{1}{2}mx^2+cx\right]_a^b,$$

for the straight line function $y = mx + c$, and this is called a **definite integral** .

Example

Find $A(x)$ for the straight line $y = 3x - 5$. Use it to work out the area under $y = 3x - 5$ between $x = 2$ and $x = 10$.

Solution

Comparing $y = 3x - 5$ with $mx + c$, you see that $m = 3$ and $c = -5$.

Hence

$$A(x) = \frac{3x^2}{2} - 5x$$

$$\text{Area} = \left[\frac{3x^2}{2} - 5x\right]_2^{10}$$

$$= \left(\frac{3\times10^2}{2} - 5\times10\right) - \left(\frac{3\times2^2}{2} - 5\times2\right)$$

$$= 100 - (-4) = 104 \text{ units}$$

Activity 4

(a) Write down an indefinite integral of the function $12 - 8x$.

(b) Evaluate the area under the graph of $y = 12 - 8x$ between $x = 1$ and $x = 3$.

(c) Draw a sketch graph and interpret your answer.

Activity 5

If $y = 2x - 7$ then the indefinite integral is $A(x) = x^2 - 7x$.

Find $\frac{dA}{dx}$. Investigate further for different straight line formulae.

Exercise 12C

1. Find the area function A for these straight lines :

   (a) $y = 2x + 7$

   (b) $s = 10 - t$

   (c) $z = 2.8 + 11.4w$

   (d) $y = -14 - 11x$

2. Use the area function method to calculate the following areas. Try to set out your working as in the worked example above.

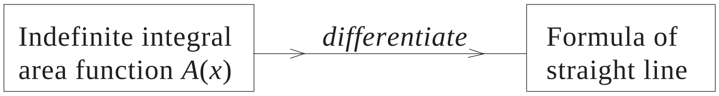
   (a) The area under $y = 6x + 1$ between $x = 0$ and $x = 3$.

   (b) The area under $s = 13 - 5t$ between $t = -2$ and $t = 1$.

   (c) $\int_1^{10} (x + 0.5)\, dx$

12.4 The reverse of differentiation

Activity 5 gives a vital clue in the search for a method of integrating more difficult functions. For straight lines, the formula of the line and its indefinite integral are connected by the following rule.

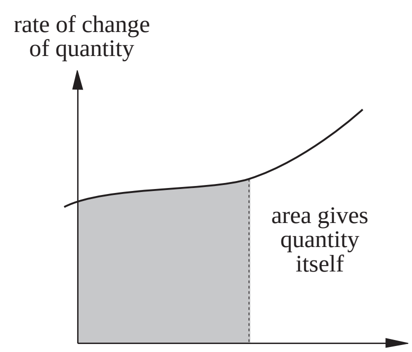


For example, in the worked example just before Activity 5, the equation of the line was $y = 3x - 5$. The corresponding indefinite integral was

$$A(x) = \frac{3x^2}{2} - 5x.$$

It is clear that in this case $\frac{dA}{dx} = y$. If this connection were true for more complex functions than straight lines, the process of finding indefinite integrals and consequently area for functions like $x^2 - 5x + 7$ would automatically be simplified.

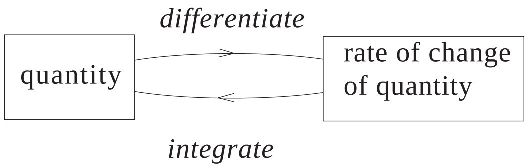
In fact, there is one important piece of evidence supporting just such a conclusion. In Section 12.1 the population was estimated by considering the area under the graph of **rate of change** of population. In Activity 3 the distance travelled by a car was calculated by finding the area under the graph of **rate of change** of distance (i.e. velocity).



In general you should appreciate that

> the area under a graph showing the **rate of change** of some quantity will give the quantity itself.

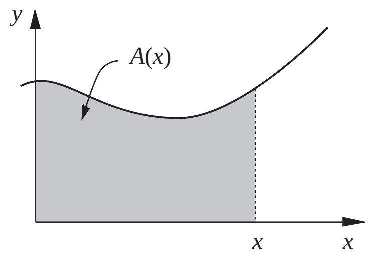
But the process of finding rates of change is differentiation, hence integration must be the reverse process.



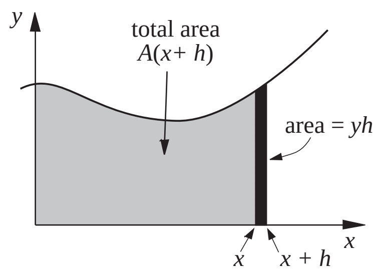
For any function, therefore, it is said that

> integration is the reverse of differentiation.

You may feel that the justification for this conclusion is rather vague: too many words and not enough mathematics! This is the way mathematicians often operate, get an instinctive 'feel' for a result and then prove it rigorously. So here is a more mathematical justification for this crucial result.



This graph represents y as a function of x. It does not matter at all what sort of function it is. Let $A(x)$ be the area under the graph between $x = 0$ and $x = x$.



Now suppose an extra strip is added to the area. It has width h and is shaded black in the diagram. The area of both shaded regions together is $A(x + h)$. The area of the black strip is approximately yh, since it is roughly a rectangle with height y.

Hence

$$A(x+h) \approx A(x) + yh$$

$$A(x+h) - A(x) \approx yh$$

$$y \approx \frac{A(x+h) - A(x)}{h}.$$

This is only approximately true (hence the '$\approx$') but the equation becomes more and more exact the smaller h becomes. But as h approaches 0, using the limit definition of differentiation,

$$\frac{A(x+h) - A(x)}{h} \to \frac{dA}{dx}.$$

Hence it can be seen that $y = \dfrac{dA}{dx}$ for any function y.

Example

What is an indefinite integral of the following?

(a) $y = 3x^2$ (b) $y = x^2$

Solution

To answer (a) consider which function, when differentiated, gives the function $3x^2$. The answer is x^3, since

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

Part (b) follows from this; x^2 is one third of $3x^2$. Hence the indefinite integral for x^2 must be

$$\frac{x^3}{3}, \text{ since } \frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2.$$

The usual way of writing the two answers above, using $\int$ notation, is

$$\int 3x^2 dx = x^3 \text{ and } \int x^2 dx = \tfrac{1}{3}x^3.$$

Activity 6 Standard functions

Following the procedure of the example above, work out these indefinite integrals :

(a) $\int x^3 dx$, $\int x^4 dx$, $\int x^5 dx$, etc

(b) $\int x^{-2} dx$, $\int x^{-3} dx$, $\int x^{-4} dx$ etc

(c) $\int x^{-1} dx$

(d) $\int e^x dx$.

Can you formulate general rules for finding indefinite integrals before turning the page and going on to the next section? Carefully consider the pattern emerging each time.

Is integration unique?

You already know that $\int 3x^2\, dx = x^3$, and that in reverse this is equivalent to saying $\frac{d}{dx}\left(x^3\right) = 3x^2$.

But what about $\frac{d}{dx}\left(x^3+1\right)$, $\frac{d}{dx}\left(x^3+4\right)$ and $\frac{d}{dx}\left(x^3-7\right)$?

They all give the answer $3x^2$. So, in general, you can write

$$\int 3x^2\, dx = x^3 + K$$

where K is any constant.

Activity 7 Integral of a constant

Sketch the graph of $y = 2$. What is $\int 2\,dx$?

Generalise to find $\int k\,dx$, where k is any number.

Standard integrals

Below is a summary of what you should have found out from Activities 6 and 7. These results are most important and, although readily available, should be memorised, so that they can be recalled instantly when needed.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + K \text{ for any integer } n, \text{ except } -1.$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \log x + K$$

$$\int e^x\,dx = e^x + K$$

Note that each of the results includes the term '+ K '. K is known as the **arbitrary** constant, or constant of integration. You must **always** include the arbitrary constant when working out **indefinite** integrals.

Examples

(1) $\int 5x^3\,dx = 5\int x^3\,dx = 5\frac{x^4}{4} + K = \frac{5x^4}{4} + K.$

(2) $\int \frac{7}{x^3}\,dx = 7\int x^{-3}\,dx$ (because $\frac{7}{x^3} = 7x^{-3}$)

$$= 7\frac{x^{-2}}{(-2)} + K = -\frac{7}{2}x^{-2} + K = -\frac{7}{2}\cdot\frac{1}{x^2} + K$$

$$= -\frac{7}{2x^2} + K.$$

(3) $\displaystyle\int\left(3x^5-\frac{2}{x}\right)dx=3\int x^5\,dx-2\int\frac{1}{x}\,dx$

$$=3\frac{x^6}{6}-2\ln x+K$$

$$=\frac{x^6}{2}-2\ln x+K$$

(4) $\displaystyle\int\frac{x+x^2}{3}\,dx=\frac{1}{3}\int\left(x+x^2\right)dx$

$$=\frac{1}{3}\left\{\int x\,dx+\int x^2\,dx\right\}$$

$$=\frac{1}{3}\left(\frac{x^2}{2}+\frac{x^3}{3}\right)+K$$

$$=\frac{x^2}{6}+\frac{x^3}{9}\quad+K$$

In the worked examples above, it has been assumed that integration is a **linear** process. For example, in Example (1), the first step was

$$\int 5x^3\,dx=5\int x^3\,dx.$$

In Example (4) the second step assumed that

$$\int\left(x+x^2\right)dx=\int x\,dx+\int x^2\,dx.$$

Why are these assumptions valid?

If $u(x)$ and $v(x)$ are any functions of x and a and b are any constant numbers, then

$$\boxed{\int\left(au(x)+bv(x)\right)dx=a\int u(x)dx+b\int v(x)dx}$$

Exercise 12D

Evaluate these indefinite integrals.
Numbers 1 to 10 are relatively straightforward.
Numbers 11 to 20 might need more care.
Tidy each answer as much as you can; for example

$-\dfrac{3}{4x^4}$ is better than $3\dfrac{x^{-4}}{(-4)}$.

1. $\int x^9\,dx$
2. $\int x^{-8}\,dx$
3. $\int \frac{1}{x^5}\,dx$
4. $\int \left(x^4+x^7\right)dx$
5. $\int \left(\frac{1}{x}+\frac{1}{x^2}\right)dx$
6. $\int 6x^7\,dx$
7. $\int \frac{3}{t^3}\,dt$
8. $\int \frac{2}{w}\,dw$
9. $\int \left(e^p-3p\right)dp$
10. $\int \left(2-\frac{1}{q^3}\right)dq$
11. $\int \frac{x^3}{2}\,dx$
12. $\int \frac{4}{3}x^7\,dx$
13. $\int \frac{3}{2y^3}\,dy$
14. $\int 2\left(x-\frac{1}{3x}\right)dx$
15. $\int \left(\frac{e^k}{4}+k^{-3}\right)dk$
16. $\int \frac{3-2x^5}{4}\,dx$
17. $\int \left(e^m-\frac{2}{5m^4}\right)dm$
18. $\int \frac{5x^2-x+1}{2}\,dx$
19. $\int \frac{1}{5}\left(z^2-\frac{1}{z}\right)dz$
20. $\int \frac{2x-1}{x}\,dx$

12.5 Finding areas

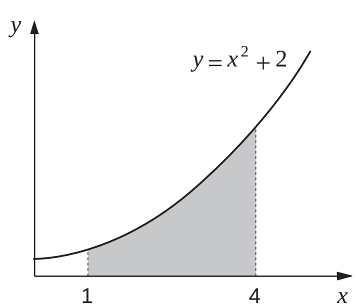
The original aim of this chapter was to calculate exact values for areas under curves. The groundwork for this has now been done. The procedure is best explained through a worked example, but before you read through it you might like to remind yourself how indefinite integrals were used to work out areas under straight lines in Section 12.2.

Example

Calculate the area under the curve $y=x^2+2$ between $x=1$ and $x=4$.

Solution

The diagram shows the area required.



Using integral notation this area would be denoted

$$\int_1^4 \left(x^2+2\right)dx.$$

The indefinite integral of x^2+2 is the function

$$\frac{x^3}{3}+2x+K$$

Hence

$$\text{Area} = \left[\frac{x^3}{3} + 2x + K\right]_1^4$$

$$= \left(\frac{4^3}{3} + 2\times 4 + K\right) - \left(\frac{1^3}{3} + 2\times 1 + K\right)$$

$$= \left(29\tfrac{1}{3} + K\right) - \left(2\tfrac{1}{3} + K\right)$$

$$= 27 \text{ units.}$$

An important point to note in this calculation is that the arbitrary constant **cancels** out. Integrals with limits, such as

$$\int_1^4 \left(x^2 + 2\right) dx$$

are called **definite** integrals, and it is customary when evaluating these to omit the arbitrary constant altogether.

Example

Work out the area under the graph of $y = 10e^x + 3x$ between $x = -1$ and $x = 3$, to one decimal place.

Solution

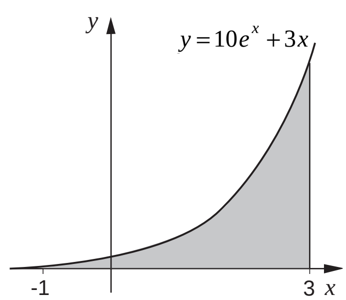
$$\text{Area} = \int_{-1}^{3} \left(10e^x + 3x\right) dx$$

$$= \left[10e^x + \frac{3x^2}{2}\right]_{-1}^{3}$$

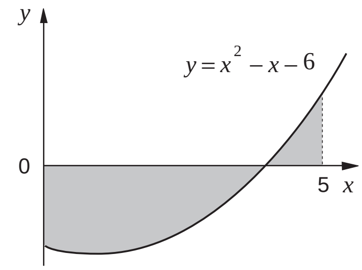
$$= \left(10e^3 + \frac{3\times 3^2}{2}\right) - \left(10e^{-1} + \frac{3\times(-1)^2}{2}\right)$$

$$= 214.35537 - 5.1787944$$

$$= 209.2 \text{ to 1 d.p.}$$



Activity 8 Negative areas



(a) Evaluate $\int_0^6 \left(x^2 - 2x - 8\right) dx$.

(b) Sketch the curve $y = x^2 - 2x - 8$ and interpret the value of the integral.

(c) Calculate the total shaded area in this diagram.

Activity 9 Simple products

Is it true that

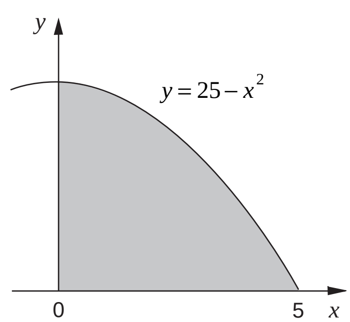
$$\int (x+3)\,(x+5)\;dx = \left(\int (x+3)\;dx\right) \times \left(\int (x+5)\;dx\right)?$$

What is the best way to evaluate $\int (x+3)(x+5)\;dx$?

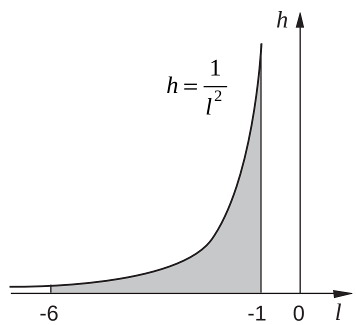
Exercise 12E

1. Work out the shaded areas below. Use integral notation when setting out your solutions.

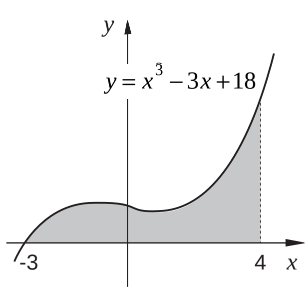
(a)



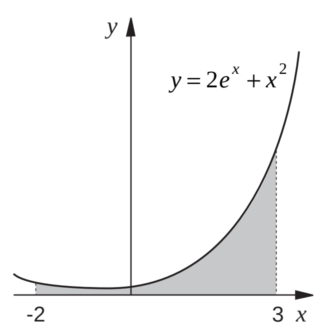
(c)



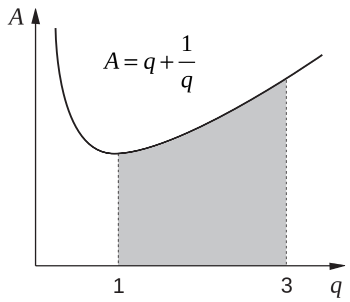
(b)



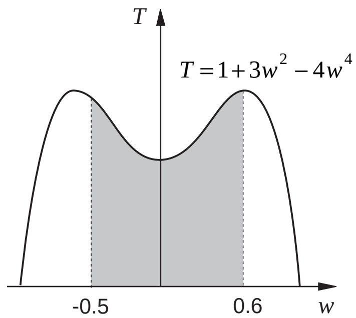
(d)



(e)



(f)

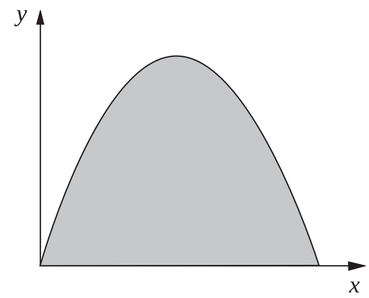


2. Evaluate these to 3 significant figures.

(a) $\int_1^5 \left(x^2 - \frac{1}{x^2} \right) dx$ (b) $\int_{-2}^{-1} \left(6z^2 - 1 \right) dz$

(c) $\int_0^1 \frac{1 + 5m^3}{6} \, dm$ (d) $\int_{2.5}^{3} \left(4 + \frac{2}{3x} \right) dx$

3. The diagram below shows a sketch of $y = x - x^2$. Find the shaded area to 3 significant figures.



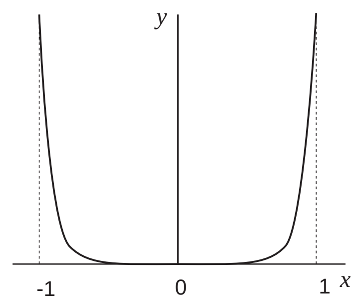
4. A pig-trough has a cross-section in the shape of the curve

$$y = x^{10},$$

for x between -1 and $+1$.

(a) Calculate $\int_{-1}^{1} x^{10} \, dx$.

(b) Work out the cross-sectional area of the trough, given that one unit on the graph represents one metre.

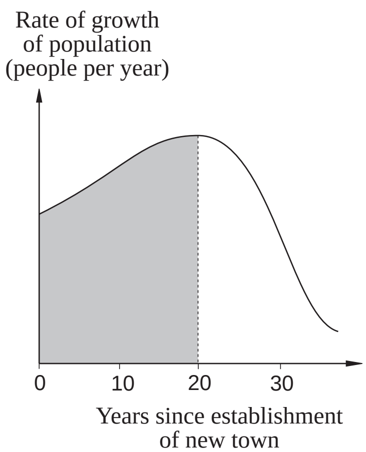


12.6 Using integration

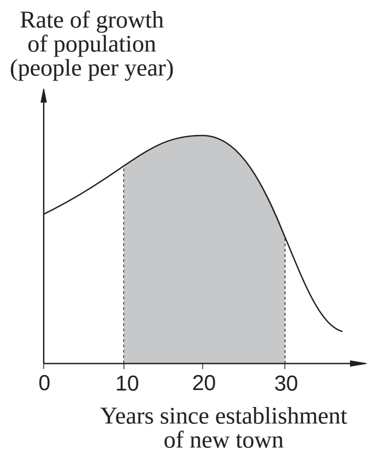
This chapter began with illustrations of particular situations where the area under a graph had significance. Activity 10 should remind you of some of them and introduce you to some more.

Activity 10 What does the area mean?

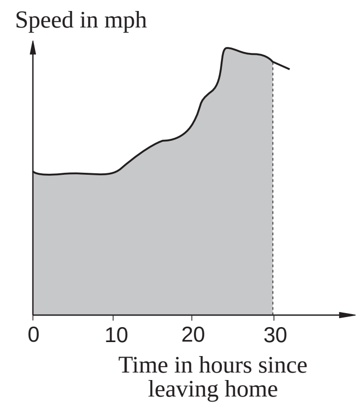
For each of these graphs, describe in words what quantity is represented by the shaded area.



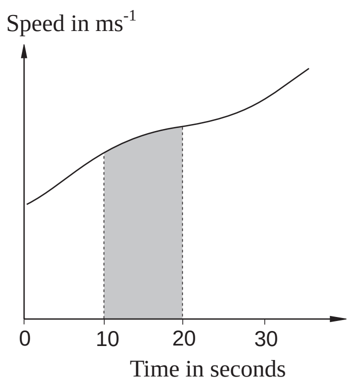
A



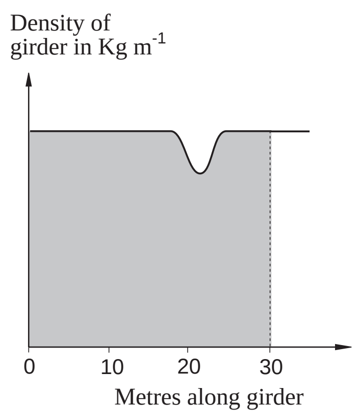
B



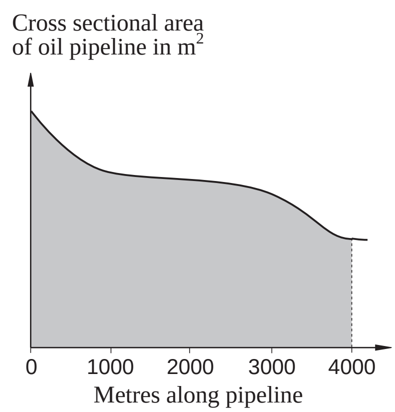
C



D



E



F

Activity 11 The rogue car

A motorist parks his car next to a telephone box. He makes a call from the box but, during the conversation, suddenly sees the car rolling down the hill; he clearly forgot to apply the handbrake. When he first noticed the car it was 20 metres away moving with speed $0.7 + 0.2t$, where t is the time in seconds after the first time he noticed the car rolling away.

(a) How far does the car travel between $t = 0$ and $t = 5$?

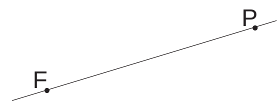
(b) Find a formula for the car's distance from the phone box, in terms of t.

12.7 Initial conditions

Activity 11 gave an example of how integration can be used to find a formula, not just a numerical value. To find the answer to part (b) properly, you needed to apply what are called initial conditions. Integrating the velocity formula produced a constant, which could be evaluated by knowing the distance when t was zero. Here are two further examples.

Example

A particle P is moving along a straight line with velocity $4+6t+t^2$. When $t=0$, P is a distance of 8 metres from a fixed point F. Find an expression for the distance FP.



Solution

The distance can be found by integrating the velocity formula.

$$\text{FP} = \int (4+6t+t^2)\,dt$$

$$= 4t + 3t^2 + \frac{t^3}{3} + \text{constant}$$

FP $= 8$ when $t = 0$, so the constant $= 8$. Hence

$$\text{FP} = 4t + 3t^2 + \frac{t^3}{3} + 8.$$

Example

A town planning committee notes that the rate of growth of the town's population since 1985 has followed the formula

$$(1500 + 200t) \text{ people per year}$$

where t is the number of years since 1st January 1985. On 1st January 1992 the population was 25 000. Find a formula for the city's population valid from 1985 onwards.

Solution

$$\text{Population } = \int (1500 + 200t)\,dt$$

$$= 1500t + 100t^2 + K.$$

When $t = 7$, $P = 25\ 000$. Putting this information into the formula gives

$$25\ 000 = 10\ 500 + 4900 + K$$

$$\Rightarrow \quad K = 9600$$

$$\text{Population} = 100t^2 + 1500t + 9600.$$

Exercise 12F

1. The speed of a falling object in ms^{-1} is given by the formula $2-10t$, where t is the time in seconds. When $t=0$ its height h above the ground is 1000 m.
   (a) Find a formula for h in terms of t.
   (b) When does the object hit the ground?

2. The speed of a particle P moving along a straight line is given by the formula
$$5+t-\frac{t^2}{4}.$$
When $t=6$, the particle is at the point A.
   (a) Find a formula for the distance AP.
   (b) Verify that the particle is again at A when t is between 7.7 and 7.8 seconds.

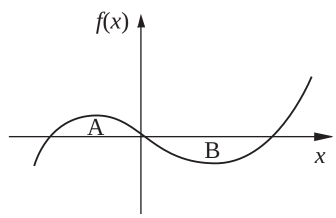
3. You have encountered the total cost function $C(Q)$ in previous questions in this text. A related economic concept is that of the marginal cost function $M(Q)$. This gives the change in total cost if the level of output (Q) is increased by 1 unit. $M(Q)$ and $C(Q)$ are thus related as follows
$$M(Q) = \frac{dC}{dQ}$$
   (a) The marginal cost function for a particular firm is $Q^2 - 3Q + 5$. Find the total cost function $C(Q)$, given that $C(0) = 5$. [$C(0)$ is the total cost when the level of output is zero, and hence gives the firm's fixed costs.]
   (b) For a different firm, $M(Q) = 2Q^2 - 10Q + 17$. Find the increase from 5 to 8 units.

12.8 Miscellaneous Exercises

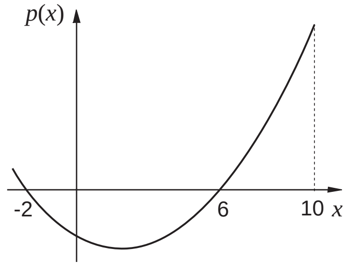
1. Work out these indefinite integrals:
   (a) $\int (5x+2)\,dx$
   (b) $\int \left(\frac{t^3}{2} + \frac{2t^2}{3}\right) dt$
   (c) $\int \left(\frac{7}{p} - p\right) dp$
   (d) $\int \frac{5}{4s^6}\,ds$
   (e) $\int \left(x - \frac{2}{x}\right)^2 dx$

2. Evaluate these definite integrals:
   (a) $\int_0^{15} (e^x + 2x^2)\,dx$
   (b) $\int_{-3}^{3} (e^x + 2x^2)\,dx$
   (c) $\int_1^2 \left(\frac{2}{x} + \frac{3}{5x^2}\right) dx$
   (d) $\int_2^3 (x+5)(x-2)\,dx$
   (e) $\int_{-2}^{-1} (x^2-2)(x+1)\,dx$

3. The graph shows the function
$$f(x) = x(x+1)(x-2).$$
Find the areas labelled A and B.

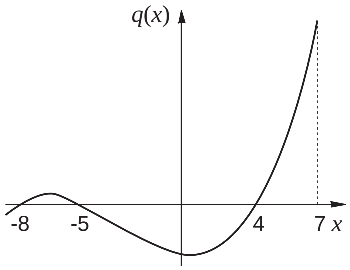


4. This graph shows a function $p(x)$.



If $\int_{-2}^{6} p(x)\,dx = -10$ and $\int_{-2}^{10} p(x)\,dx = 1$, write down the value of $\int_{6}^{10} p(x)\,dx$.

5.



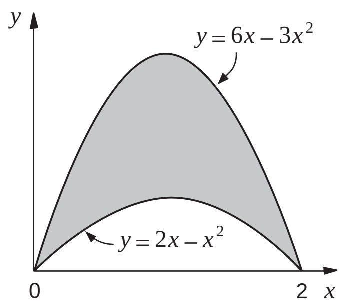
Given that

$\int_{-8}^{7} q(x)\,dx = -4, \int_{4}^{7} q(x)\,dx = 4$ and $\int_{-5}^{4} q(x)\,dx = -11,$

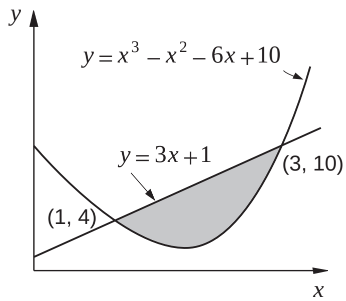
find the value of $\int_{-8}^{-5} q(x)\,dx$.

6. Find the shaded area in these diagrams.

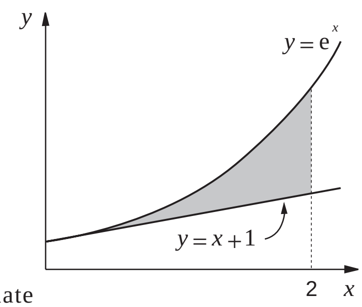
(a)



(b)



(c)



7. Evaluate

(a) $\int_{1}^{3} \dfrac{x-1}{x}\,dx$ (b) $\int_{-4}^{-1} \dfrac{xe^{x}-3}{2x}\,dx$

8. A petrol tank, when full, contains 36 litres of petrol. It develops a small hole which widens as time goes by. The rate at which fuel leaks out (in litres per day) is given by the formula

$$0.009t^2 + 0.08t + 0.01$$

where t is the time in days. When $t = 0$ the tank is full.

(a) Find the formulas for

(i) the amount of fuel lost
(ii) the amount of fuel left in the tank after t days.

(b) How many litres does the tank lose on

(i) the first day;
(ii) the tenth day?

(c) How much fuel is left in the tank after

(i) 5 days;
(ii) 15 days?

9. A gas is being kept in a large cylindrical container, the height of which can be altered by means of a piston. The pressure of the gas (p), volume in which it is kept (V) and temperature (T) are related by the equation

$$pV = 5430T.$$

(a) If $T = 293$ (degrees Kelvin) and the radius of the base of the cylinder is 1 metre, show that

$$p \approx \frac{5.06 \times 10^5}{h}$$

where h is the height of the piston above the base of the cylinder.

(b) The energy (in joules) required to compress the gas from a height h_1, to a height h_2 is given by

$$\int_{h_2}^{h_1} p\,dh$$

Initially the piston is at a height of 5 metres. How much energy is required to push the piston down to a height of

(i) 4 m (ii) 1 m?

10. $u(x)$ and $v(x)$ are two functions of x.

$$\int_0^3 u(x)\,dx = 5 \text{ and } \int_0^3 v(x)\,dx = 8.$$

Use this information to calculate these, where possible.

(a) $\int_0^3 (u(x)+v(x))dx$

(b) $\int_0^3 u(x)v(x)dx$

(c) $\int_0^3 xu(x)dx$

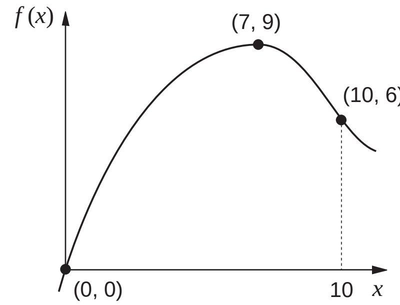
(d) $\int_0^3 (2u(x)-3v(x))\,dx$

(e) $\int_0^3 \left[(u(x))^2+(v(x))^2\right]dx$

(f) $\int_0^6 u(x)dx$

(g) $\int_0^3 \frac{u(x)}{v(x)}\,dx.$

11. The graph of a function $f(x)$ looks like this:



(a) Sketch the graphs of the following. Each sketch should have three points clearly labelled with their coordinates.

(i) $f(x)+1$ (ii) $f(x-4)$

(iii) $2f(x)$ (iv) $f(2x)$

(b) You are given

$$\int_0^{10} f(x)dx = 56.$$

Use this to calculate

(i) $\int_0^{10} [f(x)+1]dx$

(ii) $\int_0^{10} 2f(x)dx$