1 THE NATURE OF MATHEMATICS

Objectives

After studying this chapter you should

- appreciate the different aspects of mathematics;
- be able to understand how simple mathematical models are constructed to solve problems;
- understand the power and limitations of mathematical analysis;
- have revised the laws of indices.

1.0 Introduction

You are starting off on a course of mathematical study which could lead to an A-level or AS award in Mathematics. Although the **starting point** of your course is clearly defined from the work that you have done in your GCSE Mathematics course, the **end point** is not so clearly defined, since it depends on your interests and ability.

Those who are more interested in Physics and Technology will probably take further mathematical study with **Mechanics**, whilst those interested in Biology or Geography need a good grounding in techniques in **Statistics**, and those specialising in Business Studies will need further work in **Decision Mathematics**.

All these areas are mathematical and the extent of the topic can be illustrated by the fact that it is possible to take not just one A-level, but two or even three A-levels in mathematical subjects. This text is based on the current 'common core', which all A-levels in mathematics have to cover. Whilst the syllabus for the common core is quite prescriptive, the aim is to show why and how to apply the various topics under study. Of course, not all mathematical topics are immediately useful and some have historically been developed for their own sake with their applications coming later.

The first task, though, will be to see some of the ways in which mathematics is developed and used; in particular its uses

- to explain
- to predict
- to make decisions

will be illustrated in the next section.

1.1 Case studies

Bode's law

In 1772, the German astronomer, *Johann Bode*, investigated the pattern formed by the distances of planets from the sun.

At the time, only six planets were known, and the pattern he devised is shown below. The distances are measured on a scale that equates 10 units to the Sun - Earth distance.

The fit between actual distances and Bode's pattern is remarkably good.

Planet	Actual distance	Bode's pattern
Mercury	4	$0+4 = 4$
Venus	7	$3+4 = 7$
Earth	10	$6+4 = 10$
Mars	15	$12+4 = 16$
-	-	-
Jupiter	52	$48+4 = 52$
Saturn	96	$96+4 = 100$

What do you think is the missing entry?

There are also planets further out than Saturn.

Find the next two numbers in Bode's pattern.

In fact, the data continues as shown here

Planet	Actual distance
Uranus	192
Neptune	301
Pluto	395

Can you give an explanation of how Bode's Law can be adapted for this extra data?

Wind chill

When the temperature drops near zero, it is usual for weather forecasters to give both the expected air temperature, and the **wind chill** temperature - this is the temperature actually felt by someone, which depends on the wind speed and air temperature. So, for example, the wind chill temperature for an actual temperature of 0° C and wind speed of 10 mph is given by -5.5° C.

For $v > 5$ mph, the wind chill temperature is given by

$$T = 33 + \left(0.45 + 0.29\sqrt{v} - 0.02v\right)(t - 33)$$

where t° C is the air temperature and v mph the wind speed. This formula was devised by American scientists during the Second World War, and is based on experimental evidence.

Example

Find the wind chill temperature when

(a) $t = 2^\circ$ C, $v = 20$ mph;

(b) $t = 10^\circ$ C, $v = 5$ mph;

(c) $t = 0^\circ$ C, $v = 40$ mph.

Solution

(a) When $t = 2$, $v = 20$,

$$T = 33 + \left(0.45 + 0.29\sqrt{20} - 0.02 \times 20\right)(-31)$$

$$= -8.8^\circ \text{C}.$$

(b) When $t = -10$, $v = 5$,

$$T = 33 + \left(0.45 + 0.29\sqrt{5} - 0.02 \times 5\right)(-43)$$

$$= -9.9^\circ \text{C}.$$

(c) When $t = 0^\circ$ C, $v = 40$ mph,

$$T = 33 + \left(0.45 + 0.29\sqrt{40} - 0.02 \times 40\right)(-33)$$

$$= -16.0^\circ \text{C}.$$

What is the significance of a wind speed of about 5 mph?

Heptathlon

The Heptathlon is a competition for female athletes who take part in **seven** separate events (usually spread over a two day period). For each event, there is a point scoring system, based on the idea that a good competitor will score 1000 points in each event. For example, the points scoring system for the 800 m running event is

$$P = 0.11193\,(254 - m)^{1.88}$$

where m is the time taken in seconds for the athlete to run 800 m.

What points are scored for a time of 254 seconds?

Example

What points are scored for a time of 124.2 seconds, and what time would give a point score of 1000?

Solution

For $m = 124.2$,

$$P = 0.11193\,(254 - 124.2)^{1.88}$$

$$\Rightarrow \quad P = 1051.$$

(Scores are always rounded down to the nearest whole number.)

Now, to score 1000 points requires a time of m seconds where

$$1000 = 0.11193\,(254 - m)^{1.88}$$

$$\Rightarrow \quad (254 - m)^{1.88} = 8934.15$$

$$\Rightarrow \quad 254 - m = (8934.15)^{\frac{1}{1.88}}$$

$$\Rightarrow \quad m = 254 - 126.364$$

giving $\quad m = 127.64.$

All track events use a points scoring system of the form

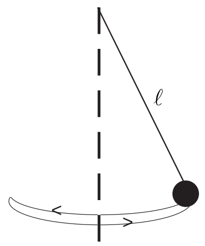
$$P = a(b - m)^c$$

with suitable constants a, b and c.

Suggest an appropriate formula for the points system in the track events in the Heptathlon.

Simple pendulum

The great Italian scientist, *Galileo*, was the first to make important discoveries about the behaviour of swinging weights. These discoveries led to the development of pendulum clocks.



Activity 1 Period of pendulum swing

Attach a weight at one end of a light string, the other end being fixed. Let the pendulum swing freely in a vertical plane and for various lengths of pendulum, ℓ, in metres, find the corresponding times in seconds of one complete oscillation (known as the **period**) - it is more accurate to time, say, five oscillations and then divide the total time by 5. On a graph, plot the period, T, against the square root of the pendulum length, $\ell^{\frac{1}{2}}$. What do you notice?

In fact, the two quantities are related by the formula

$$T = 2.006\,\ell^{\frac{1}{2}}$$

Example

What pendulum length gives a periodic time of 1 second?

Solution

If $T=1$ then

$$1 = 2.006\,\ell^{\frac{1}{2}}$$

$$\Rightarrow \quad \ell^{\frac{1}{2}} = \frac{1}{2.006} = 0.4985$$

$$\Rightarrow \quad \ell \approx 0.25\text{ m}.$$

Activity 2

Construct a simple pendulum with $\ell = 0.25$ m, and check its periodic time.

Perfect numbers

These are numbers whose divisors (excluding the number itself) add up to the number. Excluding the number 1, the first perfect number is 6, since

$$6 = 3\times 2 = 1\times 6$$

and $$3+2+1=6.$$

Activity 3

Test the numbers 7, 8, ... , 30 in order to find the next perfect number. (You might find it useful to write a short computer program to test whether any number is perfect.)

You have probably realised by now that perfect numbers are pretty thin on the ground!

Example

Are the following numbers perfect :

(a) 220 (b) 284 (c) 496?

Solution

(a)
$$\begin{aligned} 220 &= 220\times 1 \\ &= 110\times 2 \\ &= 55\times 4 \\ &= 44\times 5 \\ &= 22\times 10 \\ &= 20\times 11 \end{aligned}$$

and $1+2+4+5+10+11+20+22+44+55+110 = 284$.

Hence 220 is not a perfect number.

(b)
$$\begin{aligned} 284 &= 284\times 1 \\ &= 142\times 2 \\ &= 71\times 4 \end{aligned}$$

and $1+2+4+71+142 = 220$.

Hence 284 is not a perfect number (but note its connection with 220).

(c)
$$\begin{aligned} 496 &= 496\times 1 \\ &= 248\times 2 \\ &= 124\times 4 \\ &= 62\times 8 \\ &= 31\times 16 \end{aligned}$$

and $1+2+4+8+16+31+62+124+248 = 496$.

Hence 496 is a perfect number.

In fact 496 is the third perfect number, and 8128 is the fourth. Although there are still many unknown results concerning perfect numbers, it has been shown that

(a) all **even** perfect numbers will be of the form

$$2^{n-1}\left(2^n-1\right)$$

when n is a prime number. This number is in fact perfect when 2^n-1 is prime;

(b) all even perfect numbers end in 6 or 8;

(c) the sum of the inverses of all divisors of a perfect number add up to 2

e.g. for 6, $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\frac{1}{1}=2$.

Activity 4 Perfect numbers

Given that the fifth and sixth perfect number are 33 550 336 and 8 589 869 056 respectively, copy and complete the table below.

n	2^n-1	**prime**	$2^{n-1}\left(2^n-1\right)$	**perfect**
2	3	√	6	√
3				
5				
7				
11				
13				

You probably noticed in the example above that 220 and 284 are connected through their divisors. They are called **amicable pairs** (they are the smallest numbers that exhibit this property) and are regarded as tokens of great love. In the Bible, for example, Jacob gave Esau 220 goats to express his love (Genesis 32, verse 14).

Activity 5 Amicable pairs

Write a short program to generate amicable pairs, and use it to find the next lowest pair.

Day of the week

The algorithm below gives a method for determining the day of the week for any date this century. The date used as an example is 3 March, 1947.

1.	Write $y = \text{year}$.	$y = 1947$
2.	Evaluate $\left[\frac{y-1}{4}\right]$ ignoring the remainder.	$\left[\frac{y-1}{4}\right] = \left[\frac{1946}{4}\right] = 486$
3.	Find $D =$ day of year. (Jan. 1st = 1, ... , Feb. 1st = 32, etc.)	$D = 31+28+3 = 62$
4.	Calculate $s = y + \left[\frac{y-1}{4}\right] + D$.	$s = 1947+486+62$ $= 2495$
5.	Divide by 7 and note remainder, R.	$\frac{s}{7} = \frac{2495}{7} = 356,$ with remainder $R = 3$
6.	The remainder is the key to the day : $R = 0 \Rightarrow$ Friday; $R = 1 \Rightarrow$ Saturday, etc.	Hence 3 March 1947 was in fact a Monday.

Activity 6

(a) Use the algorithm to find the day of the week on which you were born. Note that if the year is divisible by 4, it is a leap year and February has 29 days!

(b) Analyse how and why this algorithm works.

Bar code design

Nearly all grocery products now include an identifying Bar Code on their wrapper (supermarkets now use them both for sales checkout and stock control). There are two types of EAN (European Article Numbers) - 13 digit and 8 digit. The shortened 8 digit code will be considered here. A possible example is shown opposite. The number has three parts.

0 0	3 9 9 1 0	4
↑	↑	↑
retailer's code	product code	check digit

The check digit is chosen so that

$$3\times\left(1^{st}+3^{rd}+5^{th}+7^{th}\text{ numbers}\right)+\left(2^{nd}+4^{th}+6^{th}+8^{th}\text{ numbers}\right)$$

is exactly divisible by 10. For the numbers above

$$3\times(0+3+9+0)+(0+9+1+4)$$

$$=3\times12+14=36+14=50$$

which is divisible by 10.

If the check digit is in error, the optical bar code reader will reject the code.

Example

Find the check digit for the EAN codes :

(a) $5021421x$ (b) $0042655x$.

Solution

(a) Denoting the check digit by x, the number

$$3\times(5+2+4+1)+(0+1+2+x)=3\times12+3+x=39+x$$

must be divisible by 10, so x must be 1.

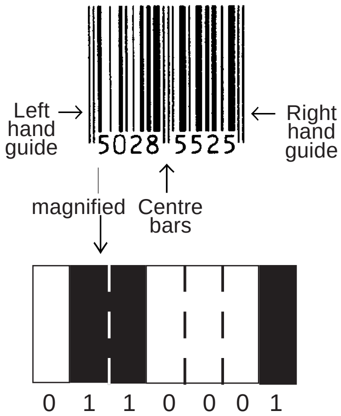
(b) Similarly

$$3\times(0+4+6+5)+(0+2+5+x)=3\times15+7+x=52+x$$

must be divisible by 10, so x must be 8.

If the optical bar code reader makes one mistake in reading a number, will it always be detected?

Another 8 digit EAN is shown opposite. It has left and right hand guide bars and centre bars. In between there are 8 bars of varying thickness. Each number is represented by a unique set of 2 bars and 2 spaces. As can be seen in the magnified version of 5, each number code is made up of 7 **modules**.



Left hand 5

The digit 5 is written as 0110001 to indicate whether a module is white (0) or black (1).

All left hand numbers start with 0 and end with 1, and use a total of 3 or 5 black modules. Right hand numbers are the complement of the corresponding left hand code e.g. right hand 5 is 1000110.

Activity 7

Design all possible codes for left hand numbers, and use 8 digit examples found on products to identify the code for each number.

The seven case studies in this section have demonstrated a variety of uses of mathematics. They have ranged from the practical design problem in Bar Codes and Bode's pattern for planetary distances to the development of perfect numbers (which as yet have no obvious applications). Mathematics embraces all these concepts, although it is the practical application side that will be emphasised where possible throughout this text. This aspect will be considered in greater depth in the next section.

Exercise 1A

1. Use the wind chill temperature formula to find its value where

   (a) $t = 0°\text{C}, \; v = 20$ mph

   (b) $t = 5°\text{C}, \; v = 20$ mph

   (c) $t = -5°\text{C}, v = 20$ mph.

   Plot a graph of wind chill temperature against air temperature, t, for v=20 mph. Use your graph to estimate the wind chill temperature when $t = 10°\text{C}$ and $v = 20$ mph.

2. The points scoring system for the high jump event in the heptathlon is given by

$$p = a(m-b)^c$$

   where $a = 1.84523$, $b = 75.0$, $c = 1.348$ and m is the height jumped in centimetres. Find the points scored for a jump of 183 cm, and determine the height required to score 1000 points.

3. An algorithm for determining the date of Easter Sunday is given at the top of the next column.

   Use it to find the date of Easter next year, which is given by the pth day of the nth month.

Step	Number	Divide by	Answer	Remainder (if needed)
1	x = year	100	$b=$	$c=$
2	$5b+c$	19	-	$a=$
3	$3(b+25)$	4	$r=$	$s=$
4	$8(b+11)$	25	$t=$	-
5	$19a+r-t$	30	-	$h=$
6	$a+11h$	319	$g=$	-
7	$60(5-s)+c$	4	$j=$	$k=$
8	$2j-k-h+g$	7	-	$m=$
9	$h-g+m+110$	30	$n=$	$q=$
10	$q+5-n$	32	-	$p=$

*4. Write a computer program to determine the date of Easter Sunday for the next 100 years. Illustrate the data using a histogram.

5. Find the check digits for these EAN codes :

   (a) 0034548* (b) 5023122*.

6. Determine whether these EAN codes have the correct check digit :

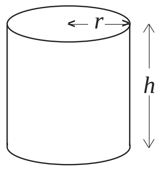
   (a) 00306678 (b) 06799205.

1.2 Applying mathematics

Mathematics can be a very powerful tool in solving practical problems. An example of this is given below with an optimisation problem of the type met in the commercial world, as well as two further case studies showing how mathematics is used to solve problems.

Metal cans

The most popular size of metal can contains a volume of about 440 ml. As they are produced in millions each week, any savings that can be made in their manufacture will prove significant. Part of the cost of making steel cans is based on the amount of material used, so it might be sensible to design a can which minimises the amount of metal used to enclose the required volume.



To analyse this problem, you must find an expression for the total surface area of a can. Suppose the cylindrical can has radius r and height h, then total surface area,

$$S = \text{curved surface area} + \text{top area} + \text{base area}.$$

What are the dimensions of the rectangle used for the curved surface area?

Assuming that no metal is wasted, an expression for the total surface area is given by

$$S = 2\pi rh + \pi r^2 + \pi r^2$$

$$\Rightarrow \quad S = 2\pi rh + 2\pi r^2. \qquad (1)$$

The formula for S shows that it is a function of two variables, r and h. But in reality it is a function of only one variable since r and h are constrained by having to enclose a specified volume.

You should be familiar with the formula for the volume of a cylindrical can:

$$V = \text{area of cross section} \times \text{height}$$

or, in this case

$$440 = \pi r^2 h. \qquad (2)$$

This equation can be used to find an expression for h which is substituted into (1) to eliminate h.

From (2)

$$h = \frac{440}{\pi r^2} \qquad (3)$$

and substituting into (1) gives

$$S = 2\pi r\left(\frac{440}{\pi r^2}\right) + 2\pi r^2$$

giving

$$\boxed{S = \frac{880}{r} + 2\pi r^2} \qquad (4)$$

What happens to S if r is very small or large? Does it make sense?

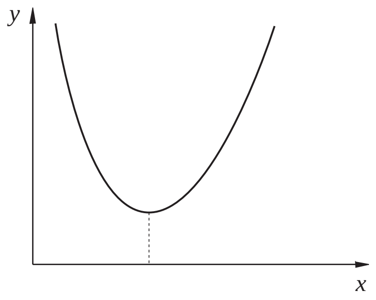
The problem is to find the value of r which minimises the total surface area S.

Activity 8 Minimising packaging costs

Draw a graph of S as given by equation (4), for x values between 1 and 10. (If you use a graphic calculator, you will need to enter the equation in the form

$$y = \frac{880}{x} + 2\pi x^2$$

where y replaces S and x replaces r.)



Use your graph to obtain an estimate of the base radius which would make the surface area of the can a minimum. (If you know how to 'magnify' a portion of the graph, you may be able to make a better estimate by concentrating on the part of the graph close to the minimum.)

Also find the corresponding optimum height from equation (3).

You have made a number of assumptions in finding this optimum value of r. To complete the analysis, you must find out whether this solution, based on minimising the total surface area, is in fact used in practice.

Activity 9 Validating the result

Find the dimensions of the usual size of a can containing 440 ml. Do they agree with your theoretical result? If not, suggest what assumptions should be modified.

In practice, the aesthetic look and feel of an object might be more important than minimising the total surface area of the packaging.

Can you suggest objects where the packaging is clearly not based on minimising the material used?

Returning to the metal can problem, there is one particular case where aesthetic appeal would not be important, and that is in the design of cans for trade use (e.g. hotels, caterers). You will see later that for these cans, your model does indeed provide the basis for deciding the optimum dimensions.

Reading age formula

Educationalists need to be able to assess the minimum reading age of certain books so that they can be appropriately catalogued, particularly for use with young children.

You are probably aware that, for example, it is much easier and quicker to read one of the tabloids (e.g. 'The Sun') than one of the quality 'heavies' (e.g. ' The Guardian').

What factors influence the reading age of a book, newspaper or pamphlet?

There have been many attempts at designing a formula for finding the reading age of a text. One example is known as the **FOG Index**. This is given by

$$R = \frac{2}{5}\left(\frac{A}{n} + \frac{100L}{A}\right) \qquad (5)$$

where the variables are defined for a sample passage of the text by

A = number of words

n = number of sentences

L = number of words containing three or more syllables (excluding '-ing' and '-ed' endings).

Activity 10

Find four or five books of varying reading difficulty. First estimate the minimum reading ages for each of these, then use the FOG formula to compare the two sets of data.

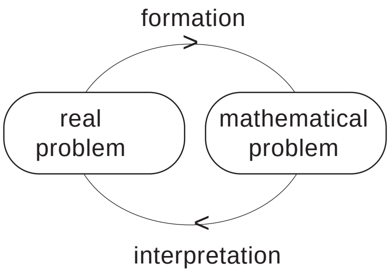
Of course, the whole concept of a designated reading age for a particular book is perhaps rather dubious. Nevertheless, the problem is a real one, and teachers and publishers do need to know the appropriate order for their reading books.

The two case studies above illustrate the idea of a **mathematical model**; that is a mathematical description of the problem. For the metal can problem the mathematical model is described by the equations (1) and (3), leading to the mathematical problem of finding the value of r which minimises

$$S = \frac{880}{r} + 2\pi r^2 .$$

In the second problem, the reading age formula, the mathematical model is essentially given by equation (5).

The translation of the problem from the real world to the mathematical world can be summarised in the diagram opposite. The model is formed from the real problem by making various assumptions, whilst the solution to the mathematical problem must be interpreted back in terms of the real problem. This will be illustrated in the next example.



Handicapping weight lifters

In weight lifting, there are nine official body weight classes. For some competitions, it is important to be able to compare lifts made by competitors in different classes. This means that some form of handicapping must be used.

There are a number of models that have been used to provide a form of handicapping. For example if

L = actual lift (in kg)

W = competitor's weight (in kg)

L' = handicapped lift,

then two possible solutions are

(a) $L' = L - W$

(b) $L' = L/(W-35)^{\frac{1}{3}}$.

The first method was used for some time in a television programme (TV Superstars) in which competitors of different weights competed against each other in a number of sports events. The second method, called the O'Carroll formula, is used in more serious competitions in order to find an overall winner.

Example

The best lifts (for the 'snatch' lift) for eight competitors are given on the following page, together with their weight. Use the two models to find an overall winner.

Competitor	Weight (in kg)	Lift (Snatch) (in kg)
1	52	105.1
2	56	117.7
3	60	125.2
4	67.5	135.2
5	75	145.2
6	82.5	162.7
7	90	170.3
8	110	175.3

Solution

The handicapped lifts are shown below.

Competitor	W	L	$L-W$	$L/(W-35)^{\frac{1}{3}}$
1	52	105.1	53.1	40.9
2	56	117.7	61.7	42.7
3	60	125.2	65.2	42.8
4	67.5	135.2	67.7	42.4
5	75	145.2	70.2	42.5
6	82.5	162.7	80.2	44.9 ←
7	90	170.3	80.3 ←	44.8
8	110	175.3	65.3	41.6

For the first method, the winner is competitor number 7, but the second method makes competitor number 6 the winner.

Because mathematics is a precise science, applications to real problems require both an understanding of the problem and an appreciation that, whilst mathematics can provide answers and give precise explanations based on particular assumptions and models, it cannot always solve the real problem. Mathematics can help to design multi-stage rockets that work, but it can't necessarily help to solve the problem of world peace. Often mathematical analysis can help in making the best decisions, and, for example the success of mathematical modelling is shown by the fact that man has stepped on the moon. You should, though, be aware that most problems in real life are more complicated than a single equation or formula!

Exercise 1B

1. Repeat the analysis for finding the minimum surface area of a metal can where the volume enclosed is 1000 ml. Determine the values of r and h which minimise the surface area.

2. A mathematical model for the reading age of a text is given by

$$R = 25 - \frac{N}{10}$$

where N is the average number of one syllable words in a passage of 150 words. Use this model to find the reading age of a number of books. Compare the results with those found in the case study outlined in Section 1.2.

3. Use the handicapping model

$$L' = L / W^{\frac{2}{3}}$$

to find the winner of the competition described in the case study in the section before this exercise.

4. Horseshoes are made by blacksmiths taking straight strips of iron and bending them into the usual horseshoe shape. To find what length of strip of iron is required, the blacksmith measures the width, W inches, of the shoe and uses a formula of the form

$$L = aW + b$$

to find the required strip length, L inches. Use the following data to find estimates for a and b:

Width W (inches)	Length L (inches)
5	12
5.75	13.50

1.3 Number

Before moving on it is important to check that you are familiar with some basic terminology and notation for numbers. Note that

(i) $\mathbb{R}$ represents the set of all real numbers

(ii) $\mathbb{Z}$ represents the set of all integers;

i.e. $\{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$

(iii) $\mathbb{N}$ represents the set of positive integers (sometimes called the natural numbers)

i.e. $\{0, 1, 2, 3, \ldots\}$

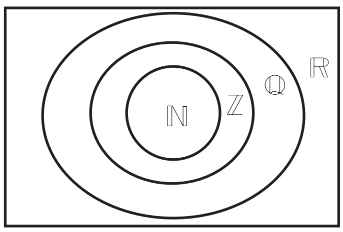
(iv) $\mathbb{Q}$ represents the set of rational numbers; that is,

numbers that can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$; for example,

$$\frac{1}{2}, \frac{1}{8}, \frac{5}{14}, \frac{21}{13}, -\frac{5}{2}, 2, \ldots$$

Why is 2 a rational number?

It is sometimes convenient to represent this system in set theory notation, by a Venn diagram of the form opposite.



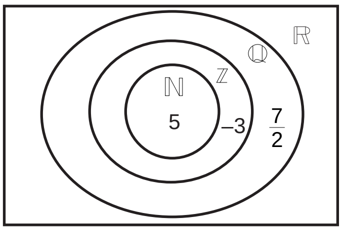
You can see that $\mathbb{N}$ is a subset of $\mathbb{Z}$, $\mathbb{Z}$ a subset of $\mathbb{Q}$, $\mathbb{Q}$ a subset of $\mathbb{R}$ or, in set theory notation

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Activity 11

Draw a Venn diagram of the type shown opposite and show which set the following numbers are in, by marking them on the diagram. The first three have already been inserted .

$$5, \frac{7}{2}, -3, -\frac{9}{4}, 7, 0, \sqrt{2}, \frac{21}{4}, -5, \pi$$



What type of numbers are $\sqrt{2}$ and π?

Any number which cannot be expressed in the form $\frac{a}{b}$ where a and b are integers ($b \neq 0$), i.e. is **not** rational, is called an **irrational** number.

Both $\sqrt{2}$ and π are irrational numbers. You will meet a proof that $\sqrt{2}$ is irrational in Section 6.8.

1.4 Indices

You have probably met the concept of indices before, but for revision purposes or in case you haven't, you should note that a^2 is shorthand notation for $a \times a$. Similarly

$$a^3 = a \times a \times a$$

$$a^4 = a \times a \times a \times a$$

and so on. Note also that

$$\begin{aligned} a^3 \times a^4 &= (a \times a \times a) \times (a \times a \times a \times a) \\ &= a \times a \times a \times a \times a \times a \times a \\ &= a^7 \end{aligned}$$

In general, you can write

$$\boxed{a^n \times a^m = a^{n+m}}$$

where n and m are positive integers. Similarly

$$\boxed{a^n \div a^m = a^{n-m}}$$

which is illustrated by

$$a^4 \div a^3 = \frac{a \times a \times a \times a}{a \times a \times a} = a$$

and here $4 - 3 = 1$.

What about $a^3 \div a^4$?

You can readily see how to evaluate this by noting that

$$a^3 \div a^4 = \frac{a^3}{a^4}$$

$$= \frac{(a \times a \times a)}{(a \times a \times a \times a)}$$

$$= \frac{1}{a}$$

But $3 - 4 = -1$, and you can write

$$a^4 \div a^3 = a^{-1};$$

that is, by definition,

$$a^{-1} = \frac{1}{a}\ .$$

Similarly

$$a^{-2} = \frac{1}{a^2} \qquad \text{etc.}$$

With this notation, the results boxed earlier are true for any integer values of m, n, and not just positive values.

Finally in this section, note that

$$\boxed{\left(a^m\right)^n = a^{mn}}$$

for integers m and n. For example, if $m = 3$ and $n = 4$,

$$\left(a^3\right)^4 = a^3 \times a^3 \times a^3 \times a^3$$

$$= (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a)$$

$$= a^{12}$$

and $mn = 3 \times 4 = 12$.

Activity 12

Verify the three boxed results when

(i) $m = 2$, $n = 3$

(ii) $m = -2$, $n = 4$

(iii) $m = -2$, $n = -3$.

One important use of indices is in **standard form**.

A number in standard form is given by

$$x \times 10^{\alpha}$$

where x is a number such that $1 \le x < 10$ and α is an **integer**.

For example

$$48.2 = 4.82 \times 10^{1}$$

$$2004 = 2.004 \times 10^{3}$$

Example

Write each of these numbers in standard form

(a) 397 (b) 76500 (c) 0.92 (d) 0.00618

Solution

(a) $397 = 3.97 \times 10^{2}$

(b) $76500 = 7.65 \times 10^{4}$

(c) $0.92 = 9.2 \times 10^{-1}$

(d) $0.00618 = 6.18 \times 10^{-3}$

It is also sometimes easier to manipulate with numbers expressed in standard form. For example

$$0.78 \times 0.021 = \left(7.8 \times 10^{-1}\right) \times \left(2.1 \times 10^{-2}\right)$$

$$= (7.8 \times 2.1) \times 10^{-1-2}$$

$$= 16.38 \times 10^{-3}$$

$$= 0.01638$$

but it is only for very large (or very small) numbers that this method is important.

Exercise 1C

1. Write in standard form

   (a) 42 (b) 0.42 (c) 0.0157

   (d) 195.2 (e) 2387.96 (f) 0.0028

2. Write as ordinary decimal numbers

   (a) 2.7×10^{2} (b) 3.52×10^{-2}

   (c) 7.01×10^{-1} (d) 5.65×10^{1}

3. Write in standard form and then find an approximate answer to

   (a) 17.2×33.1 (b) 0.49×23.8

   (c) 0.72×0.94 (d) $(0.31)^{2}$

   Use a calculator to check your answers.

4. Use a calculator to evaluate the following. Give your answers in standard form.

   (a) $\left(7.2\times10^{-2}\right)\times\left(1.6\times10^{-1}\right)$

   (b) $\left(2.74\times10^{-3}\right)\times\left(5.02\times10^{2}\right)$

   (c) $\left(1.14\times10^{-1}\right)\times\left(3.145\times10^{-1}\right)$

5. Use numbers expressed in standard form to find an approximate answer to

   (a) $98.15+0.219$ (b) $0.05128+0.477$

   Use a calculator to check your answers.

6. Light travels at a speed of 186 000 miles per second. Express the number in standard form and then find in standard form the speed of light in miles per hour.

   How far does light travel in one year, again giving your answer in standard form?

7. The mean radius of the earth is approximately 6500 km. What, in standard form, is the approximate distance round the equator?