APPENDIX : PARTIAL FRACTIONS

Given the expression $\frac{1}{x-2}-\frac{1}{x+4}$ and asked to find its integral, you can use work from Section 14.4 to give

$$\int\left(\frac{1}{x-2}-\frac{1}{x+4}\right)dx = \ln(x-2)-\ln(x+4)+c$$

$$= \ln\left(k\frac{(x-2)}{x+4}\right) \quad (c = \ln k)$$

If the same problem had been presented as

$$\int\frac{6}{x^2+2x-8}dx$$

this may have caused some difficulty.

However, since $\frac{6}{x^2+2x-8}\equiv\frac{1}{x-2}-\frac{1}{x+4}$ you can 'solve' the problem. Writing

$$\frac{6}{x^2+2x-8}\equiv\frac{1}{x-2}-\frac{1}{x+4}$$

means that you have expressed $\frac{6}{x^2+2x-8}$ in **partial fractions**.

Expressing a function of the form $\frac{1}{g(x)}$ when $g(x)$ is a polynomial in x in terms of its partial fraction is a very useful method which enables you to evaluate $\int\frac{1}{g(x)}dx$. So firstly you need to find out how to find partial fractions. The approach is illustrated in the following example.

Example

Writing

$$\frac{x-1}{(3x-5)(x-3)}=\frac{A}{(3x-5)}+\frac{B}{(x-3)}$$

find the values of A and B.

Solution

If the previous equation is true, then multiplying both sides by the denominator gives

$$x-1 = A(x-3) + B(3x-5)$$

This must hold for **any** value of x. So, for example, if $x = 3$, then

$$(3-1) = A(3-3) + B(9-5)$$

$$\Rightarrow \quad 2 = 4B$$

$$\Rightarrow \quad B = \tfrac{1}{2}$$

Similarly, $x = \tfrac{5}{3}$ gives

$$\tfrac{5}{3} - 1 = A\left(\tfrac{5}{3} - 3\right) + B(5-5)$$

$$\Rightarrow \quad \tfrac{2}{3} = -\tfrac{4}{3}A$$

$$\Rightarrow \quad A = -\tfrac{1}{2}$$

So $$\frac{x-1}{(3x-5)(x-3)} = \frac{-\tfrac{1}{2}}{(3x-5)} + \frac{\tfrac{1}{2}}{(x-3)}$$

You can check this by putting the R.H.S. over a common denominator:

$$\text{R.H.S.} = \frac{-\tfrac{1}{2}(x-3) + \tfrac{1}{2}(3x-5)}{(3x-5)(x-3)}$$

$$= \frac{-\tfrac{1}{2}x + \tfrac{3}{2} + \tfrac{3}{2}x - \tfrac{5}{2}}{(3x-5)(x-3)}$$

$$= \frac{x-1}{(3x-5)(x-3)}$$

Also note that an alternative to substituting values in the identity

$$x-1 = A(x-3) + B(3x-5)$$

is to compare coefficients. So for 'x' terms,

$$[x] \quad 1 = A + 3B$$

and for the constant term

$$[\text{ct}] \quad -1 = -3A - 5B$$

These two equations can be solved for A and B.

Check that $a = -\frac{1}{2}$, $B = \frac{1}{2}$ satisfies both equations.

Activity 1

(a) By writing $\dfrac{1}{(x+4)(x-5)}$ as $\dfrac{A}{x+4} + \dfrac{B}{x-5}$ show that

$$1 = A(x-5) + B(x+4)$$

and hence find the values of A and B.

(b) By writing $\dfrac{2x+1}{(x-4)(x+1)}$ as $\dfrac{A}{x-4} + \dfrac{B}{x+1}$ find the values of A and B and hence express $\dfrac{2x+1}{(x-4)(x+1)}$ in partial fractions.

(c) Express $\dfrac{11x+12}{(2x+3)(x+2)(x-3)}$ in partial fractions of the form

$$\frac{A}{2x+3} + \frac{B}{x+2} + \frac{C}{x-3}.$$

A quadratic expression in the denominator cannot always be expressed in terms of linear factors, for example $\dfrac{1}{x^2+1}$ or $\dfrac{2x+1}{x^2+3}$. Now $\dfrac{2x+1}{x^2+3}$ could be written as $\dfrac{2x}{x^2+3} + \dfrac{1}{x^2+3}$; this would suggest that when writing an expression where one of the factors is quadratic, there may be two unknowns to find.

For example, $\dfrac{2x+1}{(x-3)(x^2+3)}$ could be written as

$$\frac{A}{x-3} + \frac{Bx}{x^2+3} + \frac{C}{x^2+3} = \frac{A}{x-3} + \frac{Bx+C}{x^2+3}$$

and multiplying both sides by $(x-3)(x^2+3)$

gives $\quad 2x+1 = A(x^2+3) + (Bx+C)(x-3)$

or $\quad 2x+1 = x^2(A+B) + x(C-3B) + 3(A-C)$.

Activity 2

Find the values of A, B and C for the expression above.

Activity 3

Express in terms of partial fractions

$$\frac{6x^2-13}{(x-1)(x-2)\left(x^2+x+5\right)}$$

You should note the result that

$$\int \frac{2x}{x^2+a^2}\,dx = \ln\left(x^2+a^2\right)+c$$

as this will be very useful in integrating partial fractions.

Activity 4

Verify the result above by differentiating the R.H.S. and showing that it is equal to the integral.

Example

Find $\displaystyle\int \frac{(3-x)}{(x+1)\left(x^2+3\right)}\,dx$

Solution

You must first find the partial fractions by writing

$$\frac{(3-x)}{(x+1)\left(x^2+3\right)} = \frac{A}{(x+1)} + \frac{Bx+C}{\left(x^2+3\right)}$$

This gives

$$3-x = A\left(x^2+3\right)+(Bx+C)(x+1)$$

Substituting

$$x=-1 \quad \Rightarrow \quad 4=4A \quad \Rightarrow \quad A=1$$

$$\left[x^2\right] \quad \Rightarrow \quad 0=A+B \quad \Rightarrow \quad B=-1$$

$$[\text{ct}] \quad \Rightarrow \quad 3=3A+C \quad \Rightarrow \quad C=0$$

Hence

$$\int \frac{(3-x)}{(x+1)(x^2+3)}\,dx = \int \frac{1}{(x+1)} - \frac{x}{(x^2+3)}\,dx$$

$$= \ln(x+1) - \tfrac{1}{2}\ln(x^2+3) + C$$

Activity 5

Find $\int \frac{(4+3x)}{(x-3)(x^2+4)}\,dx$

One further case arises when a quadratic expression in the denominator does **not** factorise. You can regard this case as optional.

A further complication arises when there is a repeated factor. For example,

$$\frac{1}{(x+2)(x-1)^2}$$

What form will the partial fraction take?

Here you can write

$$\frac{1}{(x+2)(x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

For multiplying throughout by $(x+2)(x-1)^2$ gives

$$1 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

and, for

$$x = 1 \quad \Rightarrow \quad 1 = C.3 \quad \Rightarrow \quad C = \tfrac{1}{3}$$

$$x = -2 \quad \Rightarrow \quad 1 = A(-3)^2 \quad \Rightarrow \quad A = \tfrac{1}{9}$$

$$[x^2] \qquad 0 = A + B \quad \Rightarrow \quad B = -\tfrac{1}{9}$$

So $$\frac{1}{(x+2)(x-1)^2} = \frac{\frac{1}{9}}{(x+2)} - \frac{\frac{1}{9}}{(x-1)} + \frac{\frac{1}{3}}{(x-1)^2}$$

Example

Integrate $\frac{1}{(x+2)(x-1)^2}$

Solution

You have already seen that

$$\frac{1}{(x+2)(x-1)^2} = \frac{\frac{1}{9}}{(x+2)} - \frac{\frac{1}{9}}{(x-1)} + \frac{\frac{1}{3}}{(x-1)^2}$$

So that

$$\int \frac{1}{(x+2)(x-1)^2}\,dx = \int \frac{\frac{1}{9}}{(x+2)}\,dx - \int \frac{\frac{1}{9}}{(x-1)}\,dx + \int \frac{\frac{1}{3}}{(x-1)^2}\,dx$$

$$= \frac{1}{9}\ln(x+2) - \frac{1}{9}\ln(x-1) - \frac{1}{3}.\frac{1}{x-1} + k$$

$$= \frac{1}{9}\ln\left(\frac{x+2}{x-1}\right) - \frac{1}{3(x-1)} + k$$

(since $\ln A - \ln B = \ln\left(\frac{A}{b}\right)$)

The method can be further extended to factors of higher degree than 2. So, for example, suppose

$$f(x) = \frac{1}{(x-1)^2(x+2)^2}$$

What form will the partial fraction take?

In all the examples so far considered of the form

$$\frac{f(x)}{g(x)}$$

when f and g are both polynomials in x, it has always been the case that the degree of f is less than the degree of g. So if g is a quadratic function, the methods so far can deal with the case where f is of the form

$$f(x) = a + bx \qquad (a,\ b \text{ constants})$$

But suppose f is also a quadratic function.

What happens when both f and g are quadratic expressions?

The method will be illustrated with an example.

Example

Express $\dfrac{x^2+5}{x^2-5x+6}$ in terms of its partial fraction.

Hence find the value of

$$\int_4^5 \frac{x^2+5}{x^2-5x+6}\,dx$$

Solution

You can write

$$\frac{x^2+5}{x^2-5x+6} = \frac{x^2+5}{(x-3)(x-2)}$$

$$= A + \frac{B}{(x-3)} + \frac{C}{(x-2)}$$

Multiplying throughout by $(x-3)(x-2)$ gives

$$x^2+5 = A(x-3)(x-2) + B(x-2) + C(x-3)$$

and

$$x = 2 \quad \Rightarrow \quad 9 = C(-1) \quad \Rightarrow \quad C = -9$$
$$x = 3 \quad \Rightarrow \quad 14 = B(1) \quad \Rightarrow \quad B = 14$$
$$[x^2] \quad 1 = A \quad \Rightarrow \quad A = 1$$

Hence

$$\frac{x^2+5}{x^2-5x+6} = 1 + \frac{14}{x-3} - \frac{9}{x-2}$$

and

$$\int_4^5 \frac{x^2+5}{x^2-5x+6}\,dx = \int_4^5 1\,dx + 14\int_4^5 \frac{1}{x-3}\,dx - 9\int_4^5 \frac{1}{x-2}\,dx$$

$$= [x]_4^5 + 14[\ln(x-3)]_4^5 - 9[\ln(x-2)]_4^5$$

$$= (5-4) + 14(\ln 2 - \ln 1) - 9(\ln 3 - \ln 2)$$

$$= 1 + 23\ln 2 - 9\ln 3$$

Activity 6

Express $\frac{x^2+x+1}{(x^2-1)}$ in partial fractions and hence find

$$\int \frac{x^2+x+1}{(x^2-1)}\,dx$$

Returning to the general case of $\frac{f(x)}{g(x)}$, consider now what happens if the degree of $f(x)$ is greater than that of $g(x)$. For example,

$$\frac{x^3+x^2-1}{(x^2-4)}$$

What form will the partial fractions take for the above function?

Activity 7

Find $\int \frac{x^3+x^2-1}{(x^2-4)}\,dx$

Finally in this section it should also be noted that expressing in terms of partial fractions can be helpful in differentiation as well as integration.

Activity 8

By putting $y=\frac{x+1}{(x-2)(x+5)}$ into partial fractions, obtain

(a) $\frac{dy}{dx}$ (b) $\frac{d^2y}{dx^2}$ (c) $\frac{d^ny}{dx^n}$

Exercise

1. Express in partial fractions

   (a) $\dfrac{x}{(2-x)(1+x)}$

   (b) $\dfrac{3x-1}{(3x+1)(x-2)}$

   (c) $\dfrac{2x}{x^2+2x-3}$

   (d) $\dfrac{3}{(x-2)^2(x+2)}$

   (e) $\dfrac{2x^2-3}{x(x^2+2)}$

   (f) $\dfrac{x^2+2x}{x^2-9}$

   (g) $\dfrac{1}{(x-1)^2(x+1)}$

   (h) $\dfrac{3x}{(x+3)(x^2+1)}$

2. Evaluate $\displaystyle\int_3^4 \frac{2x-1}{(x-2)(5-x)}\,dx$

3. Evaluate $\displaystyle\int_0^{\frac{3}{4}} \frac{1-x}{(x+1)(x^2+1)}\,dx$

4. Find $\displaystyle\int \frac{2x^2+2x+3}{(x+2)(x^2+3)}\,dx$

5. Find $\displaystyle\int \frac{x^3+2x^2-10x-9}{(x-3)(x+3)}\,dx$