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Chapter 8 Differential Equations

May/June 2002

7 In a certain chemical process a substance is being formed, and t minutes after the start of the process there are m grams of the substance present. In the process the rate of increase of m is proportional to $(50 - m)^2$. When $t = 0$, $m = 0$ and $\dfrac{dm}{dt} = 5$.

  (i) Show that m satisfies the differential equation

$$\frac{dm}{dt} = 0.002(50 - m)^2.$$ [2]

  (ii) Solve the differential equation, and show that the solution can be expressed in the form

$$m = 50 - \frac{500}{t + 10}.$$ [5]

  (iii) Calculate the mass of the substance when $t = 10$, and find the time taken for the mass to increase from 0 to 45 grams.   [2]

  (iv) State what happens to the mass of the substance as t becomes very large.   [1]

Oct/Nov 2002

9 In an experiment to study the spread of a soil disease, an area of $10 \, \text{m}^2$ of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, $5 \, \text{m}^2$ was infected and the rate of growth of the infected area was $0.1 \, \text{m}^2$ per day. At time t days after the start of the experiment, an area $a \, \text{m}^2$ is infected and an area $(10 - a) \, \text{m}^2$ is uninfected.

  (i) Show that $\dfrac{da}{dt} = 0.004a(10 - a)$.   [2]

  (ii) By first expressing $\dfrac{1}{a(10 - a)}$ in partial fractions, solve this differential equation, obtaining an expression for t in terms of a.   [6]

  (iii) Find the time taken for 90% of the soil area to become infected, according to this model.   [2]

May/June 2003

7 In a chemical reaction a compound X is formed from a compound Y. The masses in grams of X and Y present at time t seconds after the start of the reaction are x and y respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of X is proportional to the mass of Y at that time. When $t = 0$, $x = 5$ and $\dfrac{dx}{dt} = 1.9$.

  (i) Show that x satisfies the differential equation

$$\frac{dx}{dt} = 0.02(100 - x).$$ [2]

  (ii) Solve this differential equation, obtaining an expression for x in terms of t.   [6]

  (iii) State what happens to the value of x as t becomes very large.   [1]

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Oct/Nov 2003

9   Compressed air is escaping from a container. The pressure of the air in the container at time t is P, and the constant atmospheric pressure of the air outside the container is A. The rate of decrease of P is proportional to the square root of the pressure difference $(P - A)$. Thus the differential equation connecting P and t is

$$\frac{dP}{dt} = -k\sqrt{(P - A)},$$

where k is a positive constant.

   (i) Find, in any form, the general solution of this differential equation.   [3]

   (ii) Given that $P = 5A$ when $t = 0$, and that $P = 2A$ when $t = 2$, show that $k = \sqrt{A}$.   [4]

   (iii) Find the value of t when $P = A$.   [2]

   (iv) Obtain an expression for P in terms of A and t.   [2]

May/June 2004

6   Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = \frac{y^3 + 1}{y^2},$$

obtaining an expression for y in terms of x.   [6]

Oct/Nov 2004

10   A rectangular reservoir has a horizontal base of area $1000 \, \text{m}^2$. At time $t = 0$, it is empty and water begins to flow into it at a constant rate of $30 \, \text{m}^3 \, \text{s}^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where h m is the depth of the water at time t s. When $h = 1$, $\frac{dh}{dt} = 0.02$.

   (i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = 0.01(3 - \sqrt{h}).$$
   [3]

It is given that, after making the substitution $x = 3 - \sqrt{h}$, the equation in part **(i)** becomes

$$(x - 3)\frac{dx}{dt} = 0.005x.$$

   (ii) Using the fact that $x = 3$ when $t = 0$, solve this differential equation, obtaining an expression for t in terms of x.   [5]

   (iii) Find the time at which the depth of water reaches $4 \, \text{m}$.   [2]

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Oct/Nov 2005

8 In a certain chemical reaction the amount, x grams, of a substance present is decreasing. The rate of decrease of x is proportional to the product of x and the time, t seconds, since the start of the reaction. Thus x and t satisfy the differential equation

$$\frac{dx}{dt} = -kxt,$$

where k is a positive constant. At the start of the reaction, when $t = 0$, $x = 100$.

(i) Solve this differential equation, obtaining a relation between x, k and t. [5]

(ii) 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

May/June 2006

5 In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container t minutes after the start of the process is x grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

(ii) Solve this differential equation, obtaining an expression for x in terms of t. [6]

Oct/Nov 2006

4 Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for y^2 in terms of x. [6]

May/June 2007

10 A model for the height, h metres, of a certain type of tree at time t years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9 - h)^{\frac{1}{3}}$. It is given that, when $t = 0$, $h = 1$ and $\frac{dh}{dt} = 0.2$.

(i) Show that h and t satisfy the differential equation

$$\frac{dh}{dt} = 0.1(9 - h)^{\frac{1}{3}}.$$ [2]

(ii) Solve this differential equation, and obtain an expression for h in terms of t. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

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Oct/Nov 2007

7   The number of insects in a population t days after the start of observations is denoted by N. The variation in the number of insects is modelled by a differential equation of the form
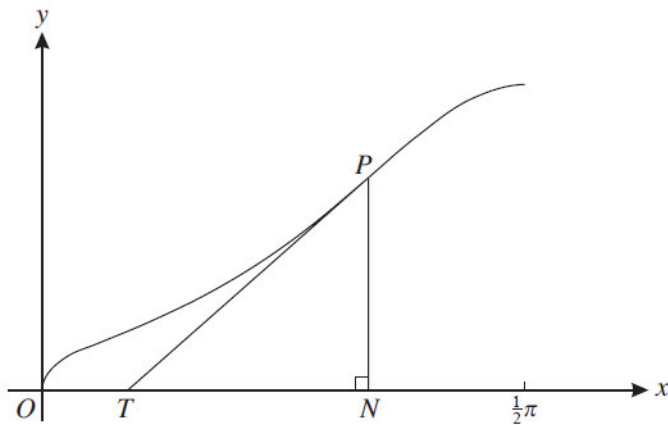
$$\frac{dN}{dt} = kN\cos(0.02t),$$

where k is a constant and N is taken to be a continuous variable. It is given that $N = 125$ when $t = 0$.

   (i) Solve the differential equation, obtaining a relation between N, k and t.   [5]

   (ii) Given also that $N = 166$ when $t = 30$, find the value of k.   [2]

   (iii) Obtain an expression for N in terms of t, and find the least value of N predicted by this model.   [3]

May/June 2008

8

In the diagram the tangent to a curve at a general point P with coordinates (x, y) meets the x-axis at T. The point N on the x-axis is such that PN is perpendicular to the x-axis. The curve is such that, for all values of x in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle PTN is equal to $\tan x$, where x is in radians.
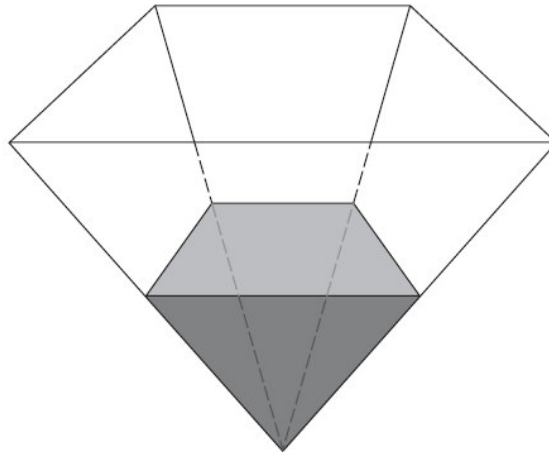
   (i) Using the fact that the gradient of the curve at P is $\dfrac{PN}{TN}$, show that

$$\frac{dy}{dx} = \tfrac{1}{2}y^2\cot x.$$ [3]

   (ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing y in terms of x.   [6]

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Oct/Nov 2008

8

An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time t hours after filling begins, the volume of liquid is $V\,\text{m}^3$ and the depth of liquid is $h\,\text{m}$. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of $20\,\text{m}^3$ per hour, but owing to leakage, liquid is lost at a rate proportional to h^2. When $h = 1$, $\dfrac{dh}{dt} = 4.95$.

(i) Show that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$ [4]

(ii) Verify that $\dfrac{20h^2}{100 - h^2} \equiv -20 + \dfrac{2000}{(10 - h)(10 + h)}$. [1]

(iii) Hence solve the differential equation in part (i), obtaining an expression for t in terms of h. [5]

May/June 2009

8 (i) Express $\dfrac{100}{x^2(10 - x)}$ in partial fractions. [4]

(ii) Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{100}x^2(10 - x),$$

obtaining an expression for t in terms of x. [6]

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Oct/Nov 2009/31

10 In a model of the expansion of a sphere of radius r cm, it is assumed that, at time t seconds after the start, the rate of increase of the surface area of the sphere is proportional to its volume. When $t = 0$, $r = 5$ and $\dfrac{dr}{dt} = 2$.

    (i) Show that r satisfies the differential equation

$$\frac{dr}{dt} = 0.08r^2.$$ [4]

    [The surface area A and volume V of a sphere of radius r are given by the formulae $A = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$.]

    (ii) Solve this differential equation, obtaining an expression for r in terms of t. [5]

    (iii) Deduce from your answer to part **(ii)** the set of values that t can take, according to this model. [1]

Oct/Nov 2009/32

9 The temperature of a quantity of liquid at time t is θ. The liquid is cooling in an atmosphere whose temperature is constant and equal to A. The rate of decrease of θ is proportional to the temperature difference $(\theta - A)$. Thus θ and t satisfy the differential equation

$$\frac{d\theta}{dt} = -k(\theta - A),$$

where k is a positive constant.

    (i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

    (ii) Given also that $\theta = 3A$ when $t = 1$, show that $k = \ln\frac{3}{2}$. [1]

    (iii) Find θ in terms of A when $t = 2$, expressing your answer in its simplest form. [3]

May/June 2010/31

5 Given that $y = 0$ when $x = 1$, solve the differential equation

$$xy\frac{dy}{dx} = y^2 + 4,$$

obtaining an expression for y^2 in terms of x. [6]

May/June 2010/32

6 The equation of a curve is

$$x \ln y = 2x + 1.$$

    (i) Show that $\dfrac{dy}{dx} = -\dfrac{y}{x^2}$. [4]

    (ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

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7 The variables x and t are related by the differential equation

$$e^{2t} \frac{dx}{dt} = \cos^2 x,$$

where $t \geqslant 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for x in terms of t. [6]

(ii) State what happens to the value of x when t becomes very large. [1]

(iii) Explain why x increases as t increases. [1]

May/June 2010/33

4 Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{x} - \frac{x}{4},$$

obtaining an expression for x^2 in terms of t. [7]

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