Chapter 5 Integration

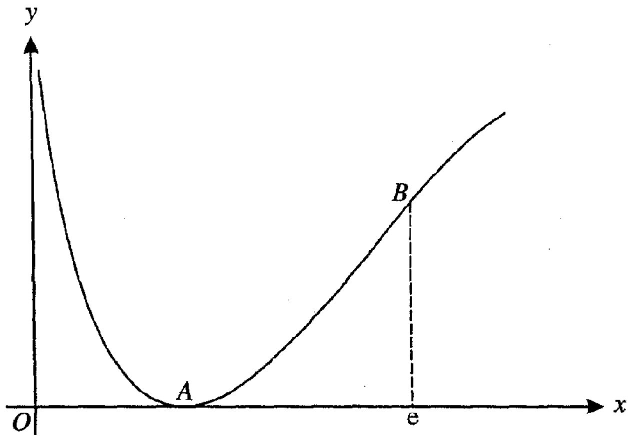
May/June 2002

6 Let $f(x) = \dfrac{4x}{(3x+1)(x+1)^2}$.

(i) Express f(x) in partial fractions. [5]

(ii) Hence show that $\int_0^1 f(x)\,dx = 1 - \ln 2$. [5]

10



The function f is defined by $f(x) = (\ln x)^2$ for $x > 0$. The diagram shows a sketch of the graph of $y = f(x)$. The minimum point of the graph is A. The point B has x-coordinate e.

(i) State the x-coordinate of A. [1]

(ii) Show that $f''(x) = 0$ at B. [4]

(iii) Use the substitution $x = e^u$ to show that the area of the region bounded by the x-axis, the line $x = e$, and the part of the curve between A and B is given by

$$\int_0^1 u^2 e^u \,du.$$ [3]

(iv) Hence, or otherwise, find the exact value of this area. [3]

Oct/Nov 2002

2 Find the exact value of $\int_1^2 x \ln x \,dx$. [4]

May/June 2003

2 Find the exact value of $\int_0^1 x e^{2x} \,dx$. [4]

10 **(i)** Prove the identity

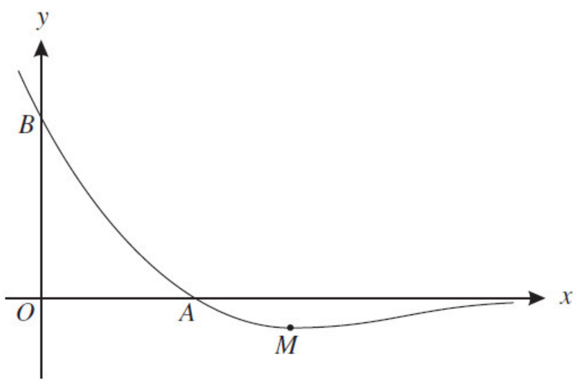
$$\cot x - \cot 2x \equiv \operatorname{cosec} 2x.$$ [3]

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx = \frac{1}{2} \ln 2$. [3]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx$, giving your answer in the form $a \ln b$. [4]

Oct/Nov 2003

6



The diagram shows the curve $y = (3 - x)e^{-2x}$ and its minimum point M. The curve intersects the x-axis at A and the y-axis at B.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of the region bounded by OA, OB and the curve, giving your answer in terms of e. [5]

8 Let $f(x) = \dfrac{x^3 - x - 2}{(x - 1)(x^2 + 1)}$.

(i) Express $f(x)$ in the form

$$A + \frac{B}{x - 1} + \frac{Cx + D}{x^2 + 1},$$

where A, B, C and D are constants. [5]

(ii) Hence show that $\int_2^3 f(x) \, dx = 1$. [4]

May/June 2004

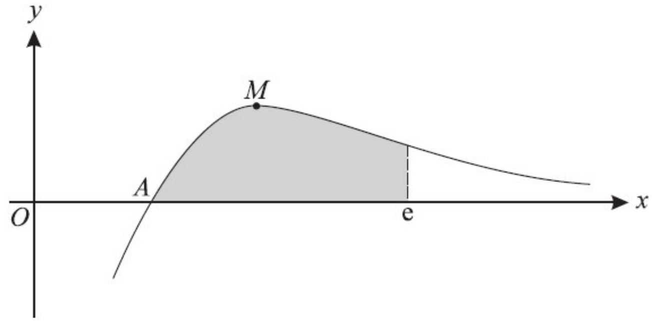
5 **(i)** Prove the identity

$$\sin^2\theta\cos^2\theta \equiv \tfrac{1}{8}(1-\cos 4\theta).$$ [3]

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta\, d\theta.$$ [3]

10



The diagram shows the curve $y = \dfrac{\ln x}{x^2}$ and its maximum point M. The curve cuts the x-axis at A.

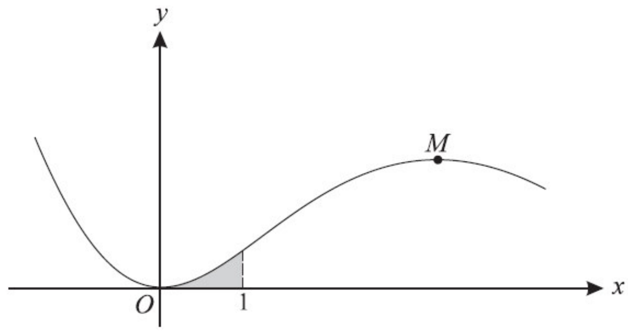
(i) Write down the x-coordinate of A. [1]

(ii) Find the exact coordinates of M. [5]

(iii) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the x-axis and the line $x = \mathrm{e}$. [5]

Oct/Nov 2004

7



The diagram shows the curve $y = x^2 e^{-\frac{1}{2}x}$.

(i) Find the x-coordinate of M, the maximum point of the curve. [4]

(ii) Find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 1$, giving your answer in terms of e. [5]

8 An appropriate form for expressing $\dfrac{3x}{(x+1)(x-2)}$ in partial fractions is

$$\frac{A}{x+1}+\frac{B}{x-2},$$

where A and B are constants.

(a) Without evaluating any constants, state appropriate forms for expressing the following in partial fractions:

(i) $\dfrac{4x}{(x+4)(x^2+3)}$, [1]

(ii) $\dfrac{2x+1}{(x-2)(x+2)^2}$. [2]

(b) Show that $\displaystyle\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx = \ln 5$. [6]

May/June 2005

4 (i) Use the substitution $x = \tan\theta$ to show that

$$\int \frac{1-x^2}{(1+x^2)^2}\,dx = \int \cos 2\theta\,d\theta.$$ [4]

(ii) Hence find the value of

$$\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx.$$ [3]

8 (i) Using partial fractions, find

$$\int \frac{1}{y(4-y)}\,dy.$$ [4]

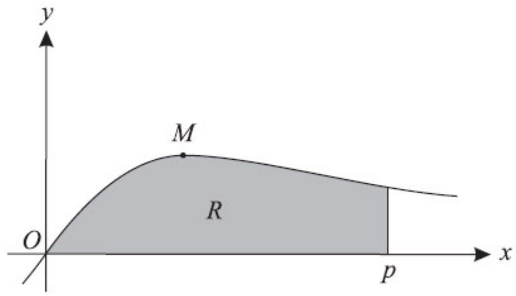
(ii) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = y(4-y),$$

obtaining an expression for y in terms of x. [4]

(iii) State what happens to the value of y if x becomes very large and positive. [1]

9



The diagram shows part of the curve $y = \frac{x}{x^2 + 1}$ and its maximum point M. The shaded region R is bounded by the curve and by the lines $y = 0$ and $x = p$.

(i) Calculate the x-coordinate of M. [4]

(ii) Find the area of R in terms of p. [3]

(iii) Hence calculate the value of p for which the area of R is 1, giving your answer correct to 3 significant figures. [2]

Oct/Nov 2005

6 **(i)** Use the substitution $x = \sin^2 \theta$ to show that

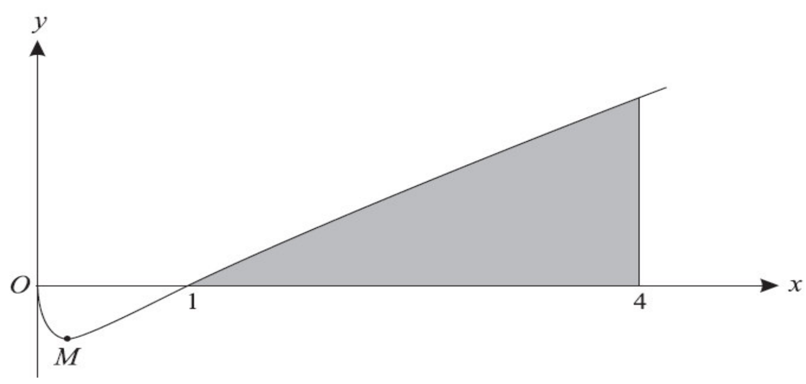
$$\int \sqrt{\left(\frac{x}{1-x}\right)}\, dx = \int 2\sin^2\theta \, d\theta. \quad [4]$$

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\, dx. \quad [4]$$

May/June 2006

8



The diagram shows a sketch of the curve $y = x^{\frac{1}{2}} \ln x$ and its minimum point M. The curve cuts the x-axis at the point $(1, 0)$.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Use integration by parts to find the area of the shaded region enclosed by the curve, the x-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

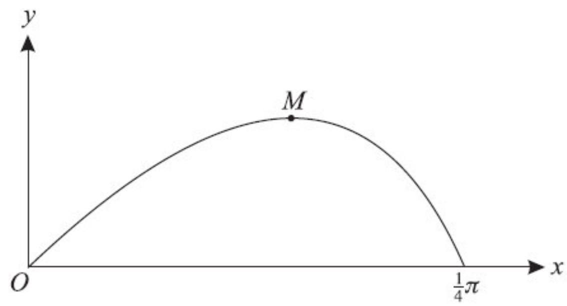
Oct/Nov 2006

8 Let $f(x) = \dfrac{7x+4}{(2x+1)(x+1)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\displaystyle\int_0^2 f(x)\,dx = 2 + \ln\tfrac{5}{3}$. [5]

10



The diagram shows the curve $y = x\cos 2x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$. The point M is a maximum point.

(i) Show that the x-coordinate of M satisfies the equation $1 = 2x\tan 2x$. [3]

(ii) The equation in part **(i)** can be rearranged in the form $x = \dfrac{1}{2}\tan^{-1}\left(\dfrac{1}{2x}\right)$. Use the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right),$$

with initial value $x_1 = 0.4$, to calculate the x-coordinate of M correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the x-axis from 0 to $\frac{1}{4}\pi$. [5]

May/June 2007

5 **(i)** Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of R and α. [3]

(ii) Hence show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{1}{\left(\cos\theta + (\sqrt{3})\sin\theta\right)^2}\,d\theta = \frac{1}{\sqrt{3}}$. [4]

7 Let $I = \displaystyle\int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx$.

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \displaystyle\int_1^2 \frac{2}{u(4-u)}\,du$. [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3$. [6]

Oct/Nov 2007

1 Find the exact value of the constant k for which $\int_1^k \frac{1}{2x-1}\,dx = 1$. [4]

3 Use integration by parts to show that

$$\int_2^4 \ln x \, dx = 6\ln 2 - 2.$$ [4]

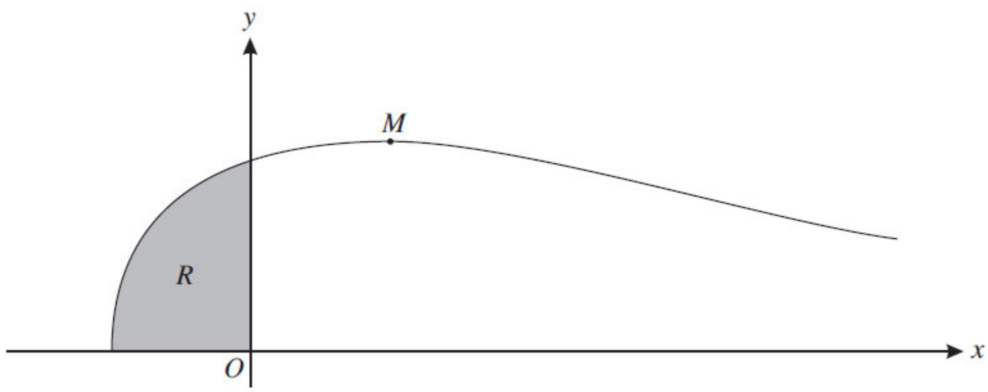
May/June 2008

7 Let $f(x) \equiv \frac{x^2 + 3x + 3}{(x+1)(x+3)}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^3 f(x)\,dx = 3 - \frac{1}{2}\ln 2$. [4]

9



The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$ and its maximum point M. The shaded region between the curve and the axes is denoted by R.

(i) Find the x-coordinate of M. [4]

(ii) Find by integration the volume of the solid obtained when R is rotated completely about the x-axis. Give your answer in terms of π and e. [6]

Oct/Nov 2008

9 The constant a is such that $\int_0^a x\mathrm{e}^{\frac{1}{2}x}\,\mathrm{d}x = 6$.

(i) Show that a satisfies the equation

$$x = 2 + \mathrm{e}^{-\frac{1}{2}x}.$$ [5]

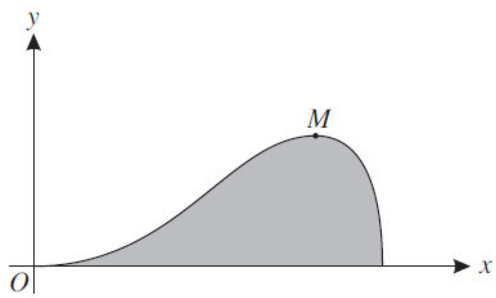
(ii) By sketching a suitable pair of graphs, show that this equation has only one root. [2]

(iii) Verify by calculation that this root lies between 2 and 2.5. [2]

(iv) Use an iterative formula based on the equation in part **(i)** to calculate the value of a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

May/June 2009

10



The diagram shows the curve $y = x^2\sqrt{(1 - x^2)}$ for $x \geqslant 0$ and its maximum point M.

(i) Find the exact value of the x-coordinate of M. [4]

(ii) Show, by means of the substitution $x = \sin\theta$, that the area A of the shaded region between the curve and the x-axis is given by

$$A = \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta\,\mathrm{d}\theta.$$ [3]

(iii) Hence obtain the exact value of A. [4]

Oct/Nov 2009/31

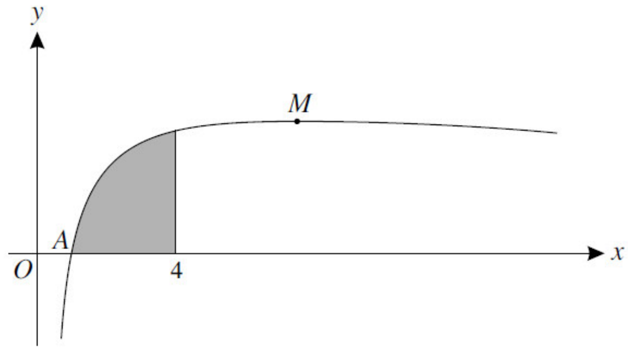
5 **(i)** Prove the identity $\cos 4\theta - 4\cos 2\theta + 3 \equiv 8\sin^4\theta$. [4]

(ii) Using this result find, in simplified form, the exact value of

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin^4\theta\,\mathrm{d}\theta.$$ [4]

7

9



The diagram shows the curve $y = \frac{\ln x}{\sqrt{x}}$ and its maximum point M. The curve cuts the x-axis at the point A.

(i) State the coordinates of A. [1]

(ii) Find the exact value of the x-coordinate of M. [4]

(iii) Using integration by parts, show that the area of the shaded region bounded by the curve, the x-axis and the line $x = 4$ is equal to $8 \ln 2 - 4$. [5]

Oct/Nov 2009/32

6 **(i)** Use the substitution $x = 2 \tan \theta$ to show that

$$\int_0^2 \frac{8}{(4 + x^2)^2}\, dx = \int_0^{\frac{1}{4}\pi} \cos^2 \theta \, d\theta.$$ [4]

(ii) Hence find the exact value of

$$\int_0^2 \frac{8}{(4 + x^2)^2}\, dx.$$ [4]

May/June 2010/31

4 **(i)** Using the expansions of $\cos(3x - x)$ and $\cos(3x + x)$, prove that

$$\tfrac{1}{2}(\cos 2x - \cos 4x) \equiv \sin 3x \sin x.$$ [3]

(ii) Hence show that

$$\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi} \sin 3x \sin x \, dx = \tfrac{1}{8}\sqrt{3}.$$ [3]

8 **(i)** Express $\dfrac{2}{(x+1)(x+3)}$ in partial fractions. [2]

(ii) Using your answer to part **(i)**, show that

$$\left(\frac{2}{(x+1)(x+3)}\right)^2 \equiv \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+3} + \frac{1}{(x+3)^2}.$$ [2]

(iii) Hence show that $\displaystyle\int_0^1 \frac{4}{(x+1)^2(x+3)^2}\,dx = \frac{7}{12} - \ln\frac{3}{2}$. [5]

May/June 2010/32

2 Show that $\displaystyle\int_0^{\pi} x^2 \sin x \, dx = \pi^2 - 4$. [5]

10 **(i)** Find the values of the constants A, B, C and D such that

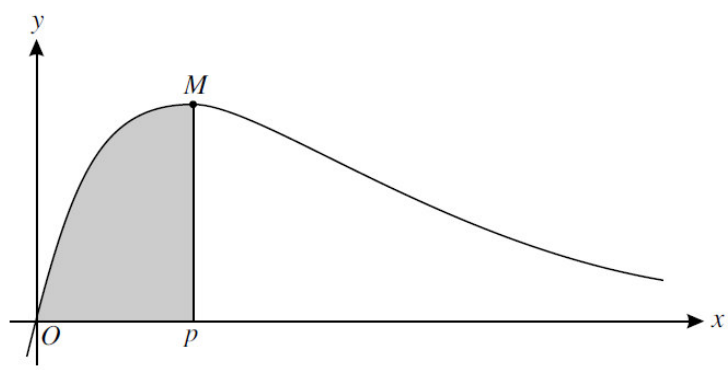
$$\frac{2x^3-1}{x^2(2x-1)} \equiv A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{2x-1}.$$ [5]

(ii) Hence show that

$$\int_1^2 \frac{2x^3-1}{x^2(2x-1)}\,dx = \frac{3}{2} + \frac{1}{2}\ln\left(\frac{16}{27}\right).$$ [5]

May/June 2010/33

5



The diagram shows the curve $y = e^{-x} - e^{-2x}$ and its maximum point M. The x-coordinate of M is denoted by p.

(i) Find the exact value of p. [4]

(ii) Show that the area of the shaded region bounded by the curve, the x-axis and the line $x = p$ is equal to $\frac{1}{8}$. [4]

7 **(i)** Prove the identity $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$. [4]

(ii) Using this result, find the exact value of

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} \cos^3\theta \, d\theta.$$ [4]