

## Chapter 4 - Vectors

The Vector Product (or Cross Product)

This is defined as

$$\underline{a} \times \underline{b} = \underline{|a| |b| \sin \theta \hat{n}}$$

where:  $\theta$  is the angle between  $\underline{a}$  and  $\underline{b}$

$\hat{n}$  is a unit vector normal to both  $\underline{a}$  and  $\underline{b}$   
(ie normal to the plane containing  $\underline{a}$  and  $\underline{b}$ )  
in the direction a screw would move if  
turned from  $\underline{a}$  to  $\underline{b}$

[The Scalar product (dot product) is defined as

$$\underline{a} \cdot \underline{b} = |a| |b| \cos \theta$$

and is a number, not a vector ]

The definition implies that :-

①  $\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$

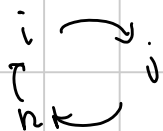
② If  $\underline{a}$  and  $\underline{b}$  are parallel,  $\underline{a} \times \underline{b} = \underline{0}$  (the zero vector)

In particular,  $\underline{a} \times \underline{a} = \underline{0}$

③ If  $|a| = |b| = 1$  and  $\underline{a} \perp \underline{b}$ , then  $\underline{a} \times \underline{b} = \hat{n}$

(a unit vector normal to  $\underline{a}$  and  $\underline{b}$ )

In particular,  $\underline{i} \times \underline{j} = \underline{k}$  (and  $\underline{j} \times \underline{i} = -\underline{k}$ )



$$\underline{j} \times \underline{k} = \underline{i}$$

$$\underline{k} \times \underline{i} = \underline{j}$$

④  $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$  (Distributive law)  
 (we will assume this)  
[www.youtube.com/megalecture](http://www.youtube.com/megalecture)  
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From the above we can find a formula for the product of 2 vectors in component form

let  $\underline{v}_1 = x_1 \underline{i} + y_1 \underline{j} + z_1 \underline{k}$   
 $\underline{v}_2 = x_2 \underline{i} + y_2 \underline{j} + z_2 \underline{k}$

Then  $\underline{v}_1 \times \underline{v}_2 = (x_1 \underline{i} + y_1 \underline{j} + z_1 \underline{k}) \times (x_2 \underline{i} + y_2 \underline{j} + z_2 \underline{k})$   
 $= \cancel{x_1 x_2 \underline{i} \times \underline{i}} + x_1 y_2 \underline{i} \times \underline{j} + x_1 z_2 \underline{i} \times \underline{k}$   
 $+ y_1 x_2 \underline{j} \times \underline{i} + \cancel{y_1 y_2 \underline{j} \times \underline{j}} + y_1 z_2 \underline{j} \times \underline{k}$   
 $+ z_1 x_2 \underline{k} \times \underline{i} + z_1 y_2 \underline{k} \times \underline{j} + \cancel{z_1 z_2 \underline{k} \times \underline{k}}$   
 $= (y_1 z_2 - z_1 y_2) \underline{i} + (z_1 x_2 - x_1 z_2) \underline{j} + (x_1 y_2 - y_1 x_2) \underline{k}$

This can be remembered by writing:—

$\underline{i}$	$\underline{j}$	$\underline{k}$	
$x_1$	$y_1$	$z_1$	— components of 1 <sup>st</sup> vector
$x_2$	$y_2$	$z_2$	— components of 2 <sup>nd</sup> vector

so that the coefficient of  $\underline{i}$  is found using the 'cross' shown in red above, and the same for  $\underline{j}$  and  $\underline{k}$  (provided we imagine the pattern to be cyclic).

Example Find  $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & -2 \\ 4 & -1 & -5 \end{vmatrix} = -17 \underline{i} - 3 \underline{j} - 13 \underline{k} \quad \text{or} \quad \begin{pmatrix} -17 \\ -3 \\ -13 \end{pmatrix}$$

# Applications of the Vector Product

① A vector equation for a line.

We usually write a line in parametric form as

$$\underline{r} = \underline{a} + \lambda \underline{v}$$

( $\underline{a}$  is the p.v. of a point on the line

$\underline{v}$  is a vector in the direction of the line

$\lambda$  is a parameter - as it varies  $\underline{r}$  is the p.v. of any point on the line)

We can also write this as

(since  $\vec{AR} \parallel \underline{v}$ )

$$(\underline{r} - \underline{a}) \times \underline{v} = \underline{0}$$

② Areas and Volumes

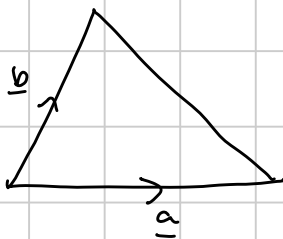
(a)



Area of parallelogram =  $|\underline{a}| |b| \sin \theta$

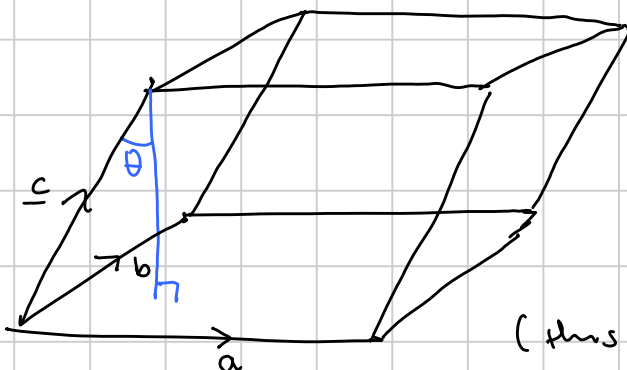
$$= |\underline{a} \times \underline{b}|$$

(b)



Area of triangle =  $\frac{1}{2} |\underline{a} \times \underline{b}|$

(c)



Volume of parallelepiped

= area of base  $\times$  perp ht

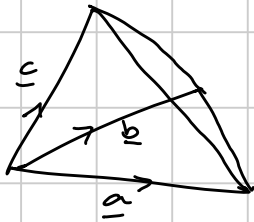
$$= |\underline{a} \times \underline{b}| |c| \cos \theta$$

$$= |(\underline{a} \times \underline{b}) \cdot \underline{c}|$$

(this last modulus is just to say ignore the sign if the result is negative)

[Note that  $(a \times b) \cdot c$  is not ambiguous because  $b \cdot c$  is a number  
 so  $a \times (b \cdot c)$  doesn't make sense.]

(d) Volume of a tetrahedron



$$\begin{aligned}
 &= \frac{1}{3} (\text{area of base}) \text{ height} \\
 &= \frac{1}{3} \left( \frac{1}{2} |a \times b| \right) |c| \cos \theta \\
 &= \frac{1}{6} a \times b \cdot c
 \end{aligned}$$

Example Find the volume of the tetrahedron with vertices at  $A(1, 1, 1)$ ,  $B(3, 2, 1)$ ,  $C(2, 1, 4)$  and  $D(3, 4, 5)$ .

$$\vec{AB} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad \vec{AD} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 3\underline{i} - 6\underline{j} - \underline{k} = \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix}$$

$$\text{So volume} = \frac{1}{6} \begin{pmatrix} 3 \\ -6 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \left| \frac{1}{6} (-16) \right| = \underline{\underline{\frac{8}{3}}}$$

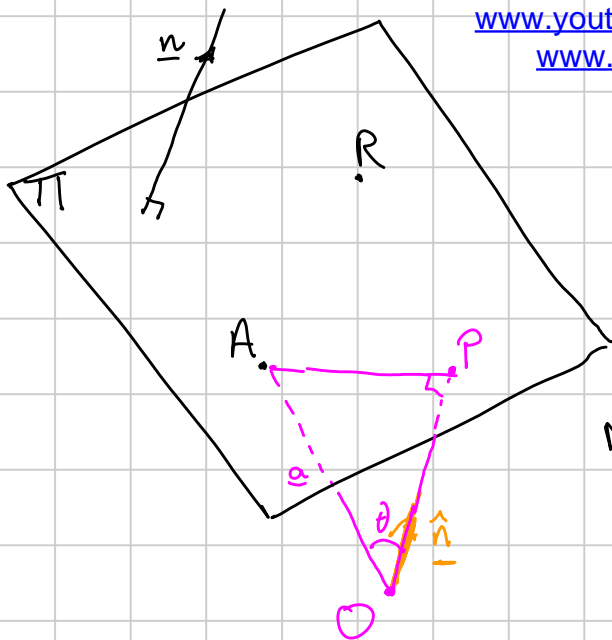
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## Equation of a Plane

The direction of a plane is defined by :

- 2 (non-parallel) vectors which lie in the plane
- or • 1 vector perpendicular to the plane

The position of a plane is fixed by the position vector of one point in the plane



Let  $\underline{n}$  be a vector normal to the plane  $\pi$ , and  $\underline{a}$  is the pv of  $A$ , a point on the plane.

Let  $\underline{r}$  be the pv of  $R$ , a general point of the plane.

Now for any position of  $R$

$$\vec{AR} \perp \underline{n}$$

or

or

$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

$$\underline{r} \cdot \underline{n} = K \text{ (a constant)}$$

This is the vector equation of a plane.

Now if we make  $\underline{n}$  a unit vector  $\hat{\underline{n}}$ , the equation above becomes:

$$\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}}$$

But  $\underline{a} \cdot \hat{\underline{n}} = |\underline{a}| \cos \theta$   
 $= OP$  on the diagram above.

which is the shortest distance from the origin to the plane. So in this form the eqn is

$$\underline{r} \cdot \hat{\underline{n}} = D$$

where  $D$  is the shortest distance from the origin to the plane

If  $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$  and  $\underline{r}$  is a general vector  $x\underline{i} + y\underline{j} + z\underline{k}$ , the equation  $\underline{r} \cdot \underline{n} = K$  becomes

$$ax + by + cz = K$$

which is the Cartesian form of the equation of a plane

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[www.youtube.com/megalecture](http://www.youtube.com/megalecture)  
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Example Find the vector eqn of  $\Pi$ , the plane normal to  $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$  passing through  $(2, -3, 1)$ .

Find the distance of this plane from the origin and the Cartesian eqn of the plane.

$$\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Vector eqn  $\underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -5 \quad (\underline{r} \cdot \underline{n} = k)$

$|\underline{n}| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6}$  so in the form  $\underline{r} \cdot \underline{\hat{n}} = D$ ,

$$\underline{r} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -\frac{5}{\sqrt{6}}$$

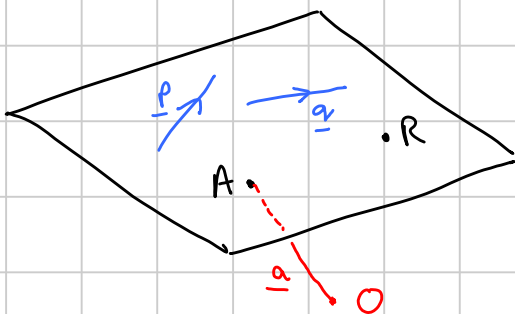
so distance from origin is  $\frac{5}{\sqrt{6}}$  units.

Cartesian eqn: let  $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

then  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -5$

$$\Rightarrow \underline{x + 2y - z = -5}$$

Alternatively, suppose we have 2 (non-parallel) vectors  $\underline{p}$  and  $\underline{q}$  which lie in the plane, and that  $\underline{a}$  is still the p.v of  $A$ , a point on the plane



Then if  $R$  is a general point on the plane,  $\underline{AR}$  can be written as  $\lambda \underline{p} + \mu \underline{q}$  (for some  $\lambda, \mu$ )

So  $\vec{OR}$  ( $= \underline{r}$ ) can be written as

$$\underline{r} = \underline{a} + \lambda \underline{p} + \mu \underline{q}$$

which is a parametric vector equation for a plane (analogous to  $\underline{r} = \underline{a} + \lambda \underline{v}$  for a line),

Also  $\underline{p} \times \underline{q}$  is a vector normal to the plane, so we can find a non-parametric equation for the plane

$$\underline{r} \cdot (\underline{p} \times \underline{q}) = \underline{a} \cdot (\underline{p} \times \underline{q})$$

Example Find a parametric equation and a vector equation for the plane containing the points A (1, 1, 2) B (2, 2, 1) and C (1, 2, 3).

$$\vec{AB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{AC} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

so a parametric eqn could be  $\underline{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 2\underline{i} - \underline{j} + \underline{k}$$

so a vector equation for the plane is

$$\underline{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 3$$

(and a Cartesian equation is  $2x - y + z = 3$ )

P127 Ex 4C Q 1a, 2a, 6a, 7a, 10b, 11, 12

Cartesian eqn of a line (in 3D)

$$r = a + \lambda v$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Leftrightarrow \left. \begin{aligned} x &= a_1 + \lambda v_1 \\ y &= a_2 + \lambda v_2 \\ z &= a_3 + \lambda v_3 \end{aligned} \right\}$$

$$\Leftrightarrow \lambda = \frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

This is the Cartesian form of the equation of a line.

Examples

① If the plane  $\Pi_1$  is  $r \cdot n_1 = c$  where  $n_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ ,  $c = 3$   
and the line  $l_1$  is  $r = a + \lambda p$  where  $a = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$ ,  $p = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

find (a) the point of intersection of  $\Pi_1$  and  $l_1$ ,  
(b) the angle between  $\Pi_1$  and  $l_1$ .

Subst eqn of line into eqn of plane: -

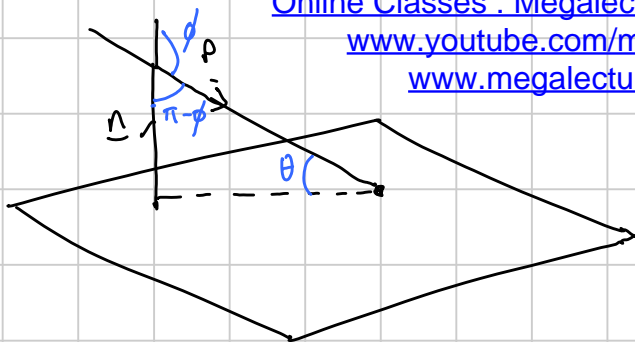
$$\left[ \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 3$$

$$\begin{aligned} -8 + \lambda(-1) &= 3 \\ \lambda &= -11 \end{aligned}$$

Subst into eqn of line: point of intersection is  $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + (-11) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$   
ie,  $(-10, 15, -19)$



(b)



$$\underline{n} \cdot \underline{p} = |\underline{n}| |\underline{p}| \cos \phi$$

$$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \sqrt{9} \sqrt{6} \cos \phi$$

$$-1 = 3\sqrt{6} \cos \phi$$

$$\cos \phi = \frac{-1}{3\sqrt{6}}$$

$$\therefore \cos(\pi - \phi) = \frac{1}{3\sqrt{6}}$$

$$\therefore \sin \theta = \frac{1}{3\sqrt{6}}$$

$$\theta = \underline{\underline{\arcsin\left(\frac{1}{3\sqrt{6}}\right)}}$$

② Given point  $S(3, -2, 1)$  and the plane  $\Pi_1$  above find:

(a) the foot of the perpendicular from  $S$  to  $\Pi_1$ ,

(b) the distance from  $S$  to  $\Pi_1$ ,

(c) the point  $S'$  which is the reflection of  $S$  in  $\Pi_1$ ,

(a) Eqn of line through  $S$   $\perp$  to  $\Pi_1$  is

$$\underline{r} = \underline{s} + \lambda \underline{n}$$

This meets the plane where

$$(\underline{s} + \lambda \underline{n}) \cdot \underline{n} = 3$$

$$\underline{s} \cdot \underline{n} + \lambda \underline{n} \cdot \underline{n} = 3$$

$$\lambda = \frac{3 - \underline{s} \cdot \underline{n}}{\underline{n} \cdot \underline{n}}$$

$$\frac{\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}}$$

$$= \frac{3 - 6}{9} = -\frac{1}{3}$$

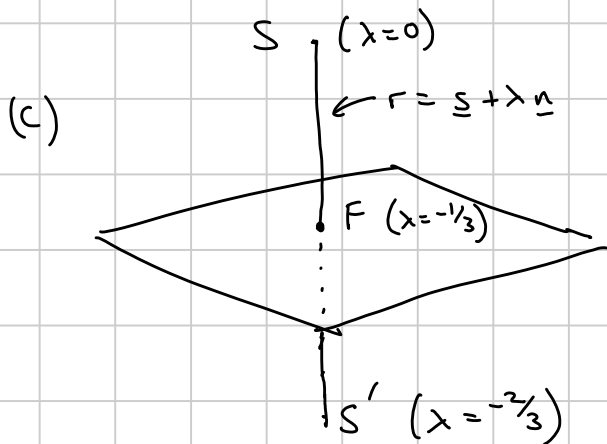
Subst into the line :  $\underline{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 7/3 \\ -5/3 \\ 5/3 \end{pmatrix}$

Point is  $(7/3, -5/3, 5/3)$

(b) Distance from point to plane =  $\left| \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 7/3 \\ -5/3 \\ 5/3 \end{pmatrix} \right|$

$$= \left| \begin{pmatrix} 2/3 \\ -1/3 \\ -2/3 \end{pmatrix} \right|$$

$$= \sqrt{4/9 + 1/9 + 4/9} = \underline{\underline{1}}$$



$$\underline{s'} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + (-2/3) \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 5/3 \\ -4/3 \\ 7/3 \end{pmatrix}$$

So  $S'$  is  $\underline{\underline{\begin{pmatrix} 5/3, -4/3, 7/3 \end{pmatrix}}}$

③ Given planes  $\Pi_1 : \underline{r} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 3$   
 and  $\Pi_2 : \underline{r} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = -1$

Find (a) their line of intersection  
 (b) the angle between them.

(a) This is best done using the Cartesian equations

$$\begin{aligned} \Pi_1 \quad 2x - y - 2z &= 3 & \textcircled{1} \\ \Pi_2 \quad x + 2y - 2z &= -1 & \textcircled{2} \end{aligned}$$

- Eliminate one variable:

$$\textcircled{1} - \textcircled{2} \Rightarrow x - 3y = 4$$

- Introduce a parameter:

$$\text{let } y = \lambda$$

- Express the other variables in terms of  $\lambda$ :

$$x = 4 + 3\lambda$$

and from  $\textcircled{1}$ ,  $2(4 + 3\lambda) - \lambda - 2z = 3$

$$8 + 6\lambda - \lambda - 2z = 3$$

$$5 + 5\lambda = 2z$$

$$z = \frac{5}{2} + \frac{5}{2}\lambda$$

$$\text{let } \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 + 3\lambda \\ \lambda \\ \frac{5}{2} + \frac{5}{2}\lambda \end{pmatrix}$$

So  $\underline{r} = \begin{pmatrix} 4 \\ 0 \\ \frac{5}{2} \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ \frac{5}{2} \end{pmatrix}$  is the line of intersection

(b) The angle between two planes is the same as the angle between their normal vectors.

$$\text{So } \underline{n}_1 \cdot \underline{n}_2 = |\underline{n}_1| |\underline{n}_2| \cos \theta$$

$$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \sqrt{9} \sqrt{9} \cos \theta$$

$$4 = 9 \cos \theta$$

$$\theta = \arccos \left( \frac{4}{9} \right)$$

(4) Find the shortest distance between lines

$$l_1 : \underline{r}_1 = \underline{a} + \lambda \underline{p} \quad \text{where } \underline{a} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \underline{p} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$l_2 : \underline{r}_2 = \underline{b} + \mu \underline{q} \quad \text{where } \underline{b} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}, \underline{q} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Let  $P_1$  and  $P_2$  be two parallel planes containing the lines



Then the shortest distance between the lines is equal to the distance between the two planes.



The vector  $\underline{p} \times \underline{q}$  is normal to both planes

$$\text{So } \underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

Plane containing  $l_1$  is  $\underline{r} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

$$\underline{r} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = 17$$

Make the normal a unit vector by dividing by  $\sqrt{(-1)^2 + 3^2 + 2^2}$

$$\underline{r} \cdot \frac{1}{\sqrt{14}} \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \frac{17}{\sqrt{14}}$$

Plane containing  $l_2$  is  $\underline{r} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

$$\underline{r} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = -9$$

$$\underline{r} \cdot \frac{1}{\sqrt{14}} \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \frac{-9}{\sqrt{14}}$$

Shortest distance between the planes (and, ∴, the lines)

$$\text{is } \frac{17}{\sqrt{14}} - \frac{-9}{\sqrt{14}} = \frac{26}{\sqrt{14}} \text{ or } \underline{\underline{\frac{13\sqrt{14}}{7}}}$$

Algebraically : [www.youtube.com/megalecture](http://www.youtube.com/megalecture) and  $\hat{n} = \frac{1}{|p \times q|} (p \times q)$

Plane containing  $l_1$  is  $r \cdot \frac{1}{|p \times q|} (p \times q) = a \cdot \frac{1}{|p \times q|} (p \times q)$

Plane containing  $l_2$  is  $r \cdot \frac{1}{|p \times q|} (p \times q) = b \cdot \frac{1}{|p \times q|} (p \times q)$

Distance between the lines is  $\frac{1}{|p \times q|} (p \times q) \cdot (a - b)$

So a condition for two lines to intersect is

$$(p \times q) \cdot (a - b) = 0$$

(and  $p \times q \neq 0$  otherwise the lines are parallel)

