

Further Vectors

Vector Product

(Remember that the scalar product $\underline{a} \cdot \underline{b}$ is a number defined as $\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta$.)

The vector product $\underline{a} \times \underline{b}$ is a vector defined as

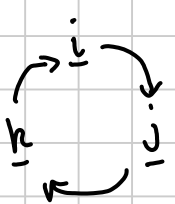
$$\underline{a} \times \underline{b} = |\underline{a}| |\underline{b}| \sin \theta \hat{n}$$

where \hat{n} is a unit vector normal to both \underline{a} and \underline{b} in the direction a screw would move if turned from \underline{a} to \underline{b} .

This definition implies that :-

- ① $\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$
- ② If \underline{a} and \underline{b} are parallel, then $\underline{a} \times \underline{b} = \underline{0}$
In particular $\underline{a} \times \underline{a} = \underline{0}$
- ③ If $|\underline{a}| = |\underline{b}| = 1$, and \underline{a} is perpendicular to \underline{b} then $\underline{a} \times \underline{b} = \hat{n}$

In particular,

	$\underline{i} \times \underline{j} = \underline{k}$ $\underline{j} \times \underline{k} = \underline{i}$ $\underline{k} \times \underline{i} = \underline{j}$	$\underline{j} \times \underline{i} = -\underline{k}$ $\underline{k} \times \underline{j} = -\underline{i}$ $\underline{i} \times \underline{k} = -\underline{j}$
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- ④ $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$ (we won't prove this)

From the above we can find a formula for the vector product of vectors in component form.

$$\begin{aligned}
 & (x_1 \underline{i} + y_1 \underline{j} + z_1 \underline{k}) \times (x_2 \underline{i} + y_2 \underline{j} + z_2 \underline{k}) \\
 &= \cancel{x_1 x_2 \underline{i} \times \underline{i}} + x_1 y_2 \underline{i} \times \underline{j} + x_1 z_2 \underline{i} \times \underline{k} \\
 &+ y_1 x_2 \underline{j} \times \underline{i} + \cancel{y_1 y_2 \underline{j} \times \underline{j}} + y_1 z_2 \underline{j} \times \underline{k} \\
 &+ z_1 x_2 \underline{k} \times \underline{i} + z_1 y_2 \underline{k} \times \underline{j} + \cancel{z_1 z_2 \underline{k} \times \underline{k}} \\
 &= \underline{(y_1 z_2 - z_1 y_2) \underline{i} + (z_1 x_2 - x_1 z_2) \underline{j} + (x_1 y_2 - y_1 x_2) \underline{k}}
 \end{aligned}$$

We can remember this by writing it in determinant form :-

$$\begin{vmatrix}
 \underline{i} & \underline{j} & \underline{k} \\
 x_1 & y_1 & z_1 \\
 x_2 & y_2 & z_2
 \end{vmatrix}$$

To find coefficient of \underline{i} , find the 2×2 determinant of the numbers which remain after crossing out.

Examples

① If $\underline{a} = \underline{i} + 2\underline{j} - 3\underline{k}$ and $\underline{b} = -\underline{i} + 2\underline{k}$, find $\underline{a} \times \underline{b}$

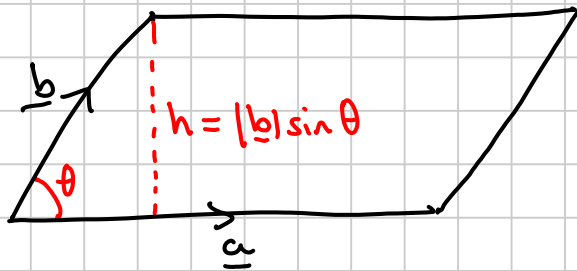
$$\begin{vmatrix}
 \underline{i} & \underline{j} & \underline{k} \\
 1 & 2 & -3 \\
 -1 & 0 & 2
 \end{vmatrix} = 4\underline{i} + \underline{j} + 2\underline{k}$$

② Simplify $(\underline{a} + \underline{b}) \times (\underline{a} - \underline{b})$

$$\begin{aligned}
 &= \underline{a} \times \underline{a} - \underline{a} \times \underline{b} + \underline{b} \times \underline{a} - \underline{b} \times \underline{b} \\
 &= \underline{0} + \underline{b} \times \underline{a} + \underline{b} \times \underline{a} - \underline{0} \\
 &= 2(\underline{b} \times \underline{a})
 \end{aligned}$$

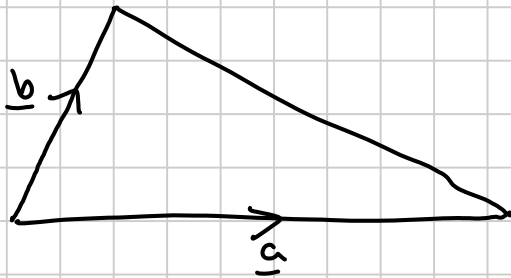
Areas and Volumes

① Parallelogram



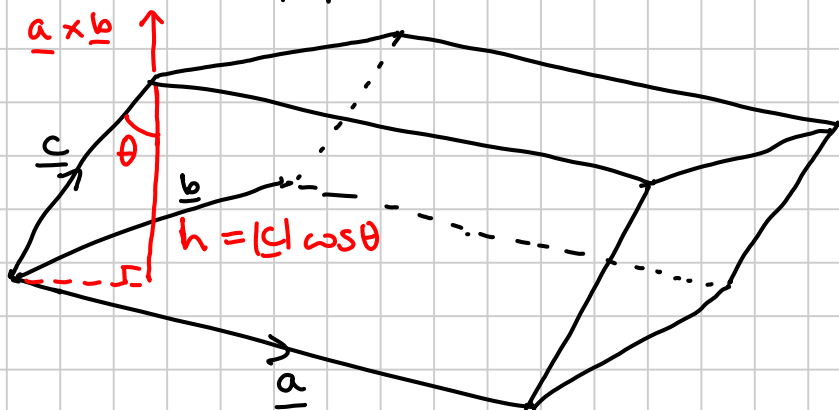
$$\begin{aligned} \text{Area} &= |\underline{a}| |\underline{b}| \sin \theta \\ &= \underline{|\underline{a} \times \underline{b}|} \end{aligned}$$

② Triangle



$$\text{Area} = \frac{1}{2} |\underline{a} \times \underline{b}|$$

③ Parallelepiped

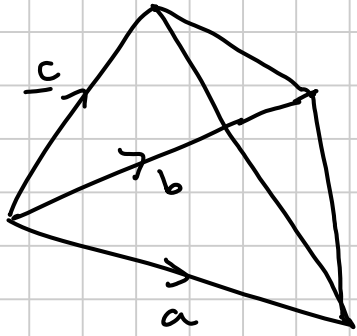


$$\begin{aligned} \text{Volume} &= \text{area of base} \times \text{perp ht} \\ &= |\underline{a} \times \underline{b}| |\underline{c}| \cos \theta \\ &= (\underline{a} \times \underline{b}) \cdot \underline{c} \end{aligned}$$

(If this is -ve, ignore the sign)

This is often written $\underline{a} \times \underline{b} \cdot \underline{c}$ since the \times must be done first otherwise we can't work out $\underline{a} \times$ (scalar)

④ Volume of a tetrahedron
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$$\begin{aligned}
 &= \frac{1}{3} (\text{area of base}) (\text{perp ht}) \\
 &= \frac{1}{3} \left(\frac{1}{2} |\underline{a} \times \underline{b}| \right) (|\underline{c}| \cos \theta) \\
 &= \underline{\underline{\frac{1}{6} (\underline{a} \times \underline{b}) \cdot \underline{c}}}
 \end{aligned}$$

p106 Ex 5A

Q 1 a gh, 2, 3, 10 a, 11

p110 Ex 5B

Q 4, 5(i), 8

Example Find the volume of the tetrahedron with vertices at P (1, 2, 1) Q (3, 3, 1) R (2, 2, 4) and S (3, 5, 5).

$$\vec{PQ} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \underline{a} \quad (\underline{q} - \underline{p})$$

$$\vec{PR} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \underline{b}$$

$$\vec{PS} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \underline{c}$$

$$\text{Volume} = \frac{1}{6} (\underline{a} \times \underline{b} \cdot \underline{c})$$

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 0 \\ 1 & 0 & 3 \end{vmatrix} = 3\underline{i} - 6\underline{j} - \underline{k}$$

$$\underline{a} \times \underline{b} \cdot \underline{c} = \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -16$$

$$\text{Volume} = \left| \frac{-16}{6} \right| = 2\frac{2}{3} \text{ cubic units}$$

P 114 Ex 5C
Q 4, 5, 6, 10

Vector product equation of a line

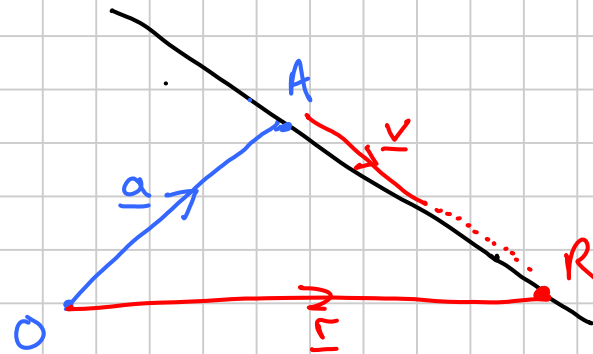
The usual equation of a line is

$$\underline{r} = \underline{a} + \lambda \underline{v}$$

An alternative way of writing this is to say that

$\underline{r} - \underline{a}$ is parallel to \underline{v}

ie/ $(\underline{r} - \underline{a}) \times \underline{v} = \underline{0}$

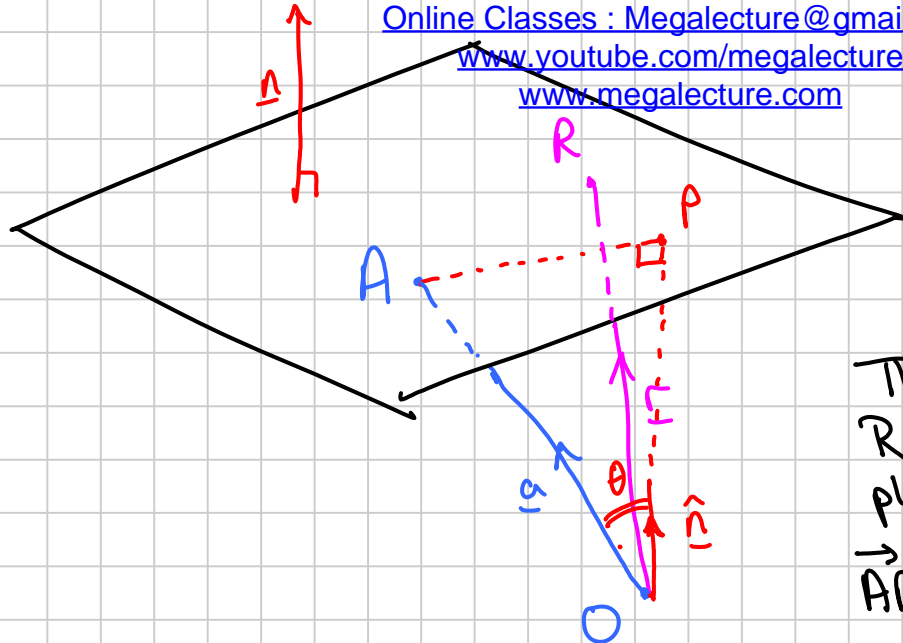


Vector equation of a plane

The DIRECTION of a plane can be defined by

- Two non-parallel vectors lying in the plane
- OR
- ONE vector normal to the plane

The POSITION of the plane can be defined by the p.v of one point in the plane.



Let \underline{n} be a normal to the plane and \underline{a} be the pv of A , a point on the plane.

Then if \underline{r} is the pv of R , a general point on the plane,

\underline{AR} is always \perp to \underline{n}

ie/
$$(\underline{r} - \underline{a}) \cdot \underline{n} = 0$$

or
$$\underline{r} \cdot \underline{n} - \underline{a} \cdot \underline{n} = 0$$

$$\underline{r} \cdot \underline{n} = \underline{a} \cdot \underline{n}$$

$$\underline{r} \cdot \underline{n} = K$$

$\leftarrow K$ is a constant

\underline{n} could be of any length. If we make it one unit long ie, $\hat{\underline{n}}$,

$$\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}}$$

$$= |\underline{a}| \cos \theta$$

$$= OP \text{ on the diagram above}$$

$$\underline{r} \cdot \hat{\underline{n}} = \text{perpendicular distance from the origin to the plane.}$$

Example (a) Find the vector equation of the plane which is normal to $\underline{i} - 3\underline{j} + \underline{k}$ and passes through the point $(2, 0, -1)$.

$$\underline{r} \cdot (\underline{i} - 3\underline{j} + \underline{k}) = (2\underline{i} - \underline{k}) \cdot (\underline{i} - 3\underline{j} + \underline{k})$$

$$\underline{r} \cdot (\underline{i} - 3\underline{j} + \underline{k}) = 1$$

(b) Hence find the distance from the origin to this plane.

Divide both sides by $\sqrt{1^2+3^2+1^2} = \sqrt{11}$

$$\therefore \frac{1}{\sqrt{11}} (\underline{i} - 3\underline{j} + \underline{k}) = \frac{1}{\sqrt{11}}$$

Distance from 0 to plane = $\frac{1}{\sqrt{11}}$

Cartesian equation for plane

If we let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$

then $\underline{r} \cdot \underline{n} = k$

becomes

$$ax + by + cz = k.$$

which is the cartesian equation

e.g. the Cartesian equation of the plane in the example above is $xc - 3y + z = 1$

Parametric Eqn of a Plane

Now suppose we have two vectors \underline{p} and \underline{q} which lie in the plane, and \underline{a} is still the pt of A, a point on the plane.

Then we can form a vector \vec{OR} where R can be any on the plane, by

$$\underline{r} = \underline{a} + \lambda \underline{p} + \mu \underline{q}$$

(this is analagous to $\underline{r} = \underline{a} + \lambda \underline{v}$ for a line)

To find a normal vector \underline{n} from \underline{p} and \underline{q} , we can work out $\underline{p} \times \underline{q}$.

Example Find a parametric equation and a vector equation for the plane containing the points A(1,1,2) B(2,2,1) and C(1,2,3)

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$$\underline{p} = \overrightarrow{AB} = \underline{i} + \underline{j} - \underline{k}$$

$$\underline{q} = \overrightarrow{AC} = \underline{j} + \underline{k}$$

Parametric eqn
$$\underline{r} = (\underline{i} + \underline{j} + 2\underline{k}) + \lambda(\underline{i} + \underline{j} - \underline{k}) + \mu(\underline{j} + \underline{k})$$

$$\underline{n} = \underline{p} \times \underline{q} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 2\underline{i} - \underline{j} + \underline{k}$$

Vector eqn
$$\underline{r} \cdot (2\underline{i} - \underline{j} + \underline{k}) = (\underline{i} + \underline{j} + 2\underline{k}) \cdot (2\underline{i} - \underline{j} + \underline{k})$$

$$\underline{r} \cdot (2\underline{i} - \underline{j} + \underline{k}) = 3$$

Cartesian eqn
$$2x - y + z = 3$$

Ex 5E Q 1a, 2a, 3ab, 4ab, 5a, 6ab

Problems involving points, lines and planes

① Plane Π has equation $\underline{r} \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$
 line L has equation $\underline{r} = (\underline{i} + 4\underline{j} + 3\underline{k}) + \lambda(\underline{i} - \underline{j} + 2\underline{k})$

(a) Find the coordinates of P , the point where L meets Π .

Subst eqn of L into eqn of Π :

$$[(1 + \lambda)\underline{i} + (4 - \lambda)\underline{j} + (3 + 2\lambda)\underline{k}] \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$$

$$2(1 + \lambda) - (4 - \lambda) - 2(3 + 2\lambda) = 3$$

$$-8 - \lambda = 3$$

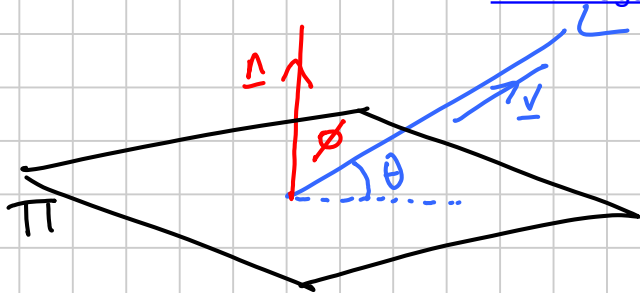
$$\lambda = -11$$

Subst $\lambda = -11$ into eqn of L :

$$\underline{r} = -10\underline{i} + 15\underline{j} - 19\underline{k}$$

Coordinates of P are $(-10, 15, -19)$

(b) Find the angle between Π and L .



p128 Ex 5F
Q 2(a,b), 6, 8

$$\underline{n} \cdot \underline{v} = |\underline{n}| |\underline{v}| \cos \phi$$

$$(2\underline{i} - \underline{j} - 2\underline{k}) \cdot (\underline{i} - \underline{j} + 2\underline{k}) = 3\sqrt{6} \cos \phi$$

$$-1 = 3\sqrt{6} \cos \phi$$

$$\cos \phi = \frac{-1}{3\sqrt{6}}$$

$$\sin \theta = \frac{-1}{3\sqrt{6}}$$

(since $\sin \theta = \cos(\frac{\pi}{2} - \theta)$)

$$\theta = \underline{\underline{\arcsin\left(\frac{-1}{3\sqrt{6}}\right)}}$$

② Π_1 is the plane $\underline{r} \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$

Π_2 is the plane $\underline{r} \cdot (\underline{i} + 2\underline{j} - 2\underline{k}) = -1$

(a) Find the equation of the line of intersection of Π_1 and Π_2 .

Write eqns in Cartesian form :

$$2x - y - 2z = 3 \quad \text{①}$$

$$x + 2y - 2z = -1 \quad \text{②}$$

Only 2 eqns with 3 unknowns, so can't solve completely (but we don't expect to do so).

• Eliminate one variable:

$$\text{①} - \text{②} \Rightarrow x - 3y = 4$$

• Introduce a parameter :

$$\begin{aligned} \text{let } y &= \lambda \\ \Rightarrow x &= 3\lambda + 4 \end{aligned}$$

- Subst back into one of the original eqns
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Eqn (1) $6x + 8 - x - 2z = 3$
 $\frac{5}{2}x + \frac{5}{2} = z$

- Eqn of line is $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$$\underline{r} = (3x+4)\underline{i} + x\underline{j} + \left(\frac{5}{2}x + \frac{5}{2}\right)\underline{k}$$

or
$$\underline{r} = \left(4\underline{i} + \frac{5}{2}\underline{k}\right) + x\left(3\underline{i} + \underline{j} + \frac{5}{2}\underline{k}\right)$$

- (b) Find the angle between Π_1 and Π_2 .

Angle between the planes = angle between normals

$$(2\underline{i} - \underline{j} - 2\underline{k}) \cdot (\underline{i} + 2\underline{j} - 2\underline{k}) = 3 \times 3 \cos \theta$$

$$4 = 9 \cos \theta$$

3ab, 4

$$\theta = \arccos\left(\frac{4}{9}\right)$$

- (3) Π is the plane $\underline{r} \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$
 S is the point $(3, -2, 1)$

- (a) Find the coordinates of F, the foot of the perpendicular from S to Π

The line through S normal to Π has equation

$$\underline{r} = (3\underline{i} - 2\underline{j} + \underline{k}) + \lambda(2\underline{i} - \underline{j} - 2\underline{k})$$

Subst into eqn of plane:

$$\left[(3+2\lambda)\underline{i} + (-2-\lambda)\underline{j} + (1-2\lambda)\underline{k} \right] \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$$

$$2(3+2\lambda) - (-2-\lambda) - 2(1-2\lambda) = 3$$

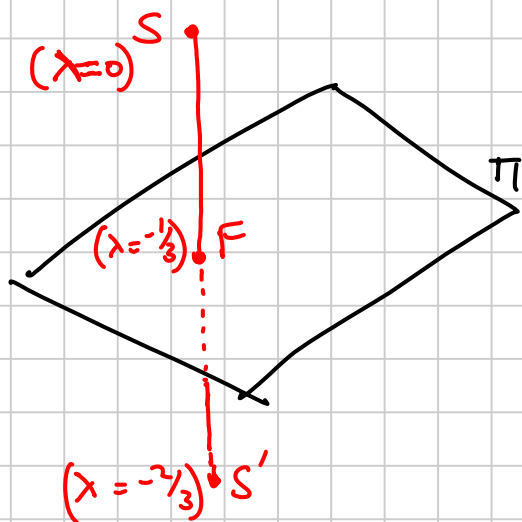
$$9\lambda + 6 = 3$$

$$\lambda = -\frac{1}{3}$$

Subst into eqn of line:

$$\underline{r} = \frac{7}{3}\underline{i} - \frac{5}{3}\underline{j} + \frac{5}{3}\underline{k} \text{ is pv of F.}$$

(ii) Find the coordinates of S' , the reflection of S in the plane Π .



Since F is the midpoint of SS' , at S' $\lambda = -2/3$

So pv of S' is

$$5/3\mathbf{i} - 4/3\mathbf{j} + 7/3\mathbf{k}$$

(iii) Find the shortest distance from S to Π .

Method 1

Find the point F as in (i).

$$\begin{aligned} \text{Shortest distance is } SF &= \sqrt{(7/3 - 3)^2 + (-5/3 + 2)^2 + (5/3 - 1)^2} \\ &= \sqrt{4/9 + 1/9 + 4/9} \\ &= \underline{\underline{1}} \end{aligned}$$

Method 2

Let Π' be a plane through S parallel to Π .

$$\begin{aligned} \text{Then } \Pi' \text{ has eqn } \underline{\underline{r}} \cdot (2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}) &= (3\underline{\underline{i}} - 2\underline{\underline{j}} + \underline{\underline{k}}) \cdot (2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}) \\ \underline{\underline{r}} \cdot (2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}) &= 6 \end{aligned}$$

$$\text{Make } \underline{\underline{n}} \text{ a unit vector } \underline{\underline{r}} \cdot \frac{1}{3}(2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}) = 2$$

So distance from the origin to Π' is 2 units.

$$\text{Now } \Pi \text{ has equation } \underline{\underline{r}} \cdot (2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}) = 3$$

$$\Rightarrow \underline{\underline{r}} \cdot \frac{1}{3}(2\underline{\underline{i}} - \underline{\underline{j}} - 2\underline{\underline{k}}) = 1$$

So distance from the origin to Π is 1 unit.

$$\text{So distance between planes} = 2 - 1 = \underline{\underline{1 \text{ unit}}}$$

④ Find the shortest distance between the lines

$$L_1 : \underline{r} = (\underline{i} + 4\underline{j} + 3\underline{k}) + \lambda (\underline{i} - \underline{j} + 2\underline{k})$$

$$L_2 : \underline{r} = (2\underline{i} - \underline{j} - 2\underline{k}) + \mu (\underline{i} + \underline{j} - \underline{k})$$

A vector normal to both lines is

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

So a plane containing L_1 is $\underline{r} \cdot (-\underline{i} + 3\underline{j} + 2\underline{k}) = (\underline{i} + 4\underline{j} + 3\underline{k}) \cdot (-\underline{i} + 3\underline{j} + 2\underline{k})$
 $\underline{r} \cdot (-\underline{i} + 3\underline{j} + 2\underline{k}) = 17$

A parallel plane containing L_2 is $\underline{r} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$
 $\underline{r} \cdot (-\underline{i} + 3\underline{j} + 2\underline{k}) = -9$

Making \underline{n} into a unit vector: $\underline{r} \cdot \frac{1}{\sqrt{14}} \underline{n} = \frac{17}{\sqrt{14}}$
 and $\underline{r} \cdot \frac{1}{\sqrt{14}} \underline{n} = \frac{-9}{\sqrt{14}}$

Distance between lines = distance between planes
 $= \frac{17 - (-9)}{\sqrt{14}} = \frac{26}{\sqrt{14}} = \underline{\underline{\frac{13\sqrt{14}}{7}}}$

⑤ Find the distance between the lines

$$L_1 : \underline{r} = (\underline{i} + 2\underline{j} - \underline{k}) + \lambda (5\underline{i} + 4\underline{j} + 3\underline{k})$$

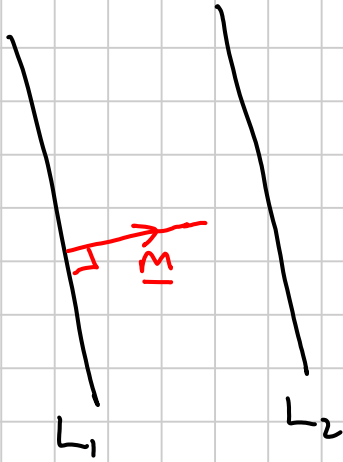
$$L_2 : \underline{r} = 2\underline{i} + \underline{k} + \mu (5\underline{i} + 4\underline{j} + 3\underline{k})$$

The vector $(2\underline{i} + \underline{k}) - (\underline{i} + 2\underline{j} - \underline{k}) = \underline{i} - 2\underline{j} + 2\underline{k}$ lies in the plane containing the two lines

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So a normal \underline{n} to this plane is $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 4 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 14\underline{i} - 7\underline{j} - 14\underline{k}$
or $2\underline{i} - \underline{j} - 2\underline{k}$

So a normal \underline{m} is $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 4 & 3 \\ 2 & -1 & -2 \end{vmatrix} = -5\underline{i} + 16\underline{j} - 13\underline{k}$



Parallel planes containing L_1 and L_2 are

$$\underline{r} \cdot \underline{m} = (\underline{i} + 2\underline{j} - \underline{k}) \cdot (-5\underline{i} + 16\underline{j} - 13\underline{k}) = 40$$

and $\underline{r} \cdot \underline{m} = (2\underline{i} + \underline{k}) \cdot (-5\underline{i} + 16\underline{j} - 13\underline{k}) = -23$

Distance between them is $\frac{40 - (-23)}{\sqrt{25 + 256 + 169}} = \frac{21\sqrt{2}}{10}$

[For an alternative method see p126 in FP3 textbook.]

p 128 Ex 5F 1ab, 11, 12, 14, 15

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Cartesian Eqn of a line in 3D
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The vector eqn of a line is $\underline{r} = \underline{a} + \lambda \underline{v}$
or $(\underline{r} - \underline{a}) \times \underline{v} = \underline{0}$

let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Then the vector eqn becomes 3 equations:

$$x = x_1 + \lambda p$$

$$y = y_1 + \lambda q$$

$$z = z_1 + \lambda r$$

Making λ the subject of each of these:

$$\lambda = \frac{x - x_1}{p} = \frac{y - y_1}{q}$$

This is the Cartesian eqn of a line.

Examples

① Write $\frac{x-2}{3} = \frac{y+4}{1} = \frac{z-1}{2}$ in vector form

let all these = λ

$$\Rightarrow x = 3\lambda + 2$$

$$y = \lambda - 4$$

$$z = 2\lambda + 1$$

let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \underline{r} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

② Write $\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ in Cartesian form

let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$

$$x = 1 + 4\lambda$$
$$y = 2$$
$$z = 3 + 5\lambda$$

($\lambda =$) $\frac{x-1}{4} = \frac{z-3}{5}$ and $y = 2$