Solving DEs using Taylor series

We have seen Taylor's series :—

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

which enables us to express any function as a power series. If x is close to a, the first few terms will be a good approximation.

[If $a=0$, this is called Maclaurin's series.]

We can use this to find approximate solutions to DEs.

An alternative notation, if $y = f(x)$, is

$$y = y_a + \left(\frac{dy}{dx}\right)_a (x-a) + \frac{1}{2!}\left(\frac{d^2y}{dx^2}\right)_a (x-a)^2 + \dots$$

Examples

① Given that $\frac{d^2y}{dx^2} = x - y$, and $y=1$ and $\frac{dy}{dx} = 2$ when $x=0$, find a series approximation for y as far as the term in x^4.

Hence find the approx value of y when $x = 0.2$.

(Here $a=0$)

$$y_0 = 1$$

$$\left(\frac{dy}{dx}\right)_0 = 2$$

$$\left(\frac{d^2y}{dx^2}\right)_0 = 0 - 1 = -1$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

$$= \frac{d}{dx}(x - y)$$

$$= 1 - \frac{dy}{dx} \qquad \text{so} \quad \left(\frac{d^3y}{dx^3}\right)_0 = 1 - 2 = -1$$

$$\frac{d^4y}{dx^4} = -\frac{d^2y}{dx^2} \qquad \text{so} \quad \left(\frac{d^4y}{dx^2}\right)_0 = -(-1) = 1$$

So $y = 1 + 2x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$

If $x = 0.2$, $\quad y \approx 1.379$

[We can solve this DE exactly to get

$$y = \cos x + \sin x + x$$

So when $x = 0.2$, $y \approx 1.379$, so the approximation is accurate to 4 s.f.]

(2) Solve the DE $\frac{dy}{dx} = \frac{2x - y}{x+1}$, given that when $x = 1$, $y = 0.5$.
Give the solution as a power series in $(x-1)$ as far as $(x-1)^3$.
Hence find an approximate value for y when $x = 1.6$

(Here $a = 1$)

$$y_1 = 0.5$$

$$\left(\frac{dy}{dx}\right)_1 = \frac{2 - 0.5}{2} = 0.75$$

Rewrite the DE to avoid using the quotient rule:

$$(x+1)\frac{dy}{dx} = 2x - y$$

Differentiate implicitly w.r.t x :

$$1\,\frac{dy}{dx} + (x+1)\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} \qquad \text{———} (*)$$

Subst values at $x = 1$

$$1 \times 0.75 + 2\left(\frac{d^2y}{dx^2}\right)_1 = 2 - 0.75$$

$$\left(\frac{d^2y}{dx^2}\right)_1 = 0.25$$

From $(*)$ $\qquad (x+1)\dfrac{d^2y}{dx^2} = 2 - 2\dfrac{dy}{dx}$

Diff : $\qquad 1.\dfrac{d^2y}{dx^2} + (x+1)\dfrac{d^3y}{dx^3} = -2\dfrac{d^2y}{dx^2}$

Subst : $\qquad 0.25 + 2\dfrac{d^3y}{dx^3} = -0.5$

$$\frac{d^3y}{dx^3} = -0.375$$

$$y = 0.5 + 0.75(x-1) + \frac{0.25}{2!}(x-1)^2 - \frac{0.375}{3!}(x-1)^3 + \dots$$

If $x = 1.6$, $y \approx 0.9815$

[Solving this DE exactly

$$\frac{dy}{dx} + \frac{y}{x+1} = \frac{2x}{x+1}$$

$$\begin{aligned} IF &= e^{\int \frac{1}{x+1}dx} \\ &= e^{\ln(x+1)} \\ &= x+1 \end{aligned}$$

$$(x+1)\frac{dy}{dx} + y = 2x$$

$$\frac{d}{dx}\left((x+1)y\right) = 2x$$

$$(x+1)y = x^2 + c$$

Subst $x=1,\ y=\frac{1}{2}$ $\qquad 2\times\frac{1}{2} = 1 + c \quad \Rightarrow \quad c = 0.$

$$y = \frac{x^2}{x+1}$$

Subst $x = 1.6 \Rightarrow \quad y \approx 0.9846$

So approximation above is accurate to 2 s f.

p135 Ex 6.4 Q 2, 3a, 4, 6, 8

p14 Ex 1C Q 7, 10, 11, 12, 16, 17