

The method of differences

This method of summing a series is best explained by an example.

Example 1

Find the sum of $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{n(n+1)}$

This is $\sum_{r=1}^n \frac{1}{r(r+1)}$, so using partial fractions write it as $\sum_{r=1}^n \left(\frac{1}{r} - \frac{1}{r+1} \right)$

$$\begin{aligned}
 \text{ie/} & \left(\frac{1}{1} - \frac{1}{2} \right) \\
 + & \left(\frac{1}{2} - \frac{1}{3} \right) \\
 + & \left(\frac{1}{3} - \frac{1}{4} \right) \\
 + & \dots \\
 + & \left(\frac{1}{n} - \frac{1}{n+1} \right)
 \end{aligned}
 \left. \vphantom{\begin{aligned} \text{ie/} \\ + \\ + \\ + \\ + \end{aligned}} \right\} \begin{array}{l} \text{now most of the terms} \\ \text{cancel out} \end{array}$$

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

The method works because we were able to write the general term of the series as $f(r) - f(r+1)$. This method (with variations) works on a variety of problems. A hint as to the method will normally be given.

Example 2

(a) Show that $r(r+1)(r+2) - (r-1)r(r+1) = 3r(r+1)$

(b) Hence find $\sum_{r=1}^n r(r+1)$

(c) Given that $\sum_{r=1}^n r = \frac{n(n+1)}{2}$, find a formula for $\sum_{r=1}^n r^2$.

(a) $r(r+1)(r+2) - (r-1)r(r+1) = r(r+1)[(r+2) - (r-1)]$
 $= \underline{\underline{3r(r+1)}}$

(b)

$$\sum_{r=1}^n r(r+1) = \frac{1}{3} \left[\sum_{r=1}^n (r+1)(r+2) - \sum_{r=1}^n (r-1)r(r+1) \right]$$

$$= \frac{1}{3} \left[\begin{aligned} & (1 \times 2 \times 3 - 0 \times 1 \times 2) \\ & + (2 \times 3 \times 4 - 1 \times 2 \times 3) \\ & + (3 \times 4 \times 5 - 2 \times 3 \times 4) \\ & + (\dots - \dots) \\ & + (n(n+1)(n+2) - (n-1)n(n+1)) \end{aligned} \right]$$

$$= \frac{1}{3} n(n+1)(n+2)$$

(c) From (b), $\sum_{r=1}^n r^2 + \sum_{r=1}^n r = \frac{1}{3} n(n+1)(n+2)$

Remember: $\sum f(r) + g(r) = \sum f(r) + \sum g(r)$

So $\sum_{r=1}^n r^2 = \frac{1}{3} n(n+1)(n+2) - \frac{1}{2} n(n+1)$

$$= \frac{1}{6} n(n+1) [2(n+2) - 3]$$

$$= \frac{1}{6} n(n+1)(2n+1)$$

Example 3 Simplify $r^2(r+1)^2 - (r-1)^2 r^2$ and hence find a formula for $\sum_{r=1}^n r^3$.

$$r^2(r+1)^2 - (r-1)^2 r^2 = r^2 [(r^2 + 2r + 1) - (r^2 - 2r + 1)] = 4r^3$$

Hence $\sum_{r=1}^n r^3 = \frac{1}{4} \sum_{r=1}^n [r^2(r+1)^2 - (r-1)^2 r^2]$

$$= \frac{1}{4} \left[\begin{aligned} & (1^2 \times 2^2 - 0^2 \times 1^2) \\ & + (2^2 \times 3^2 - 1^2 \times 2^2) \\ & + (\dots - \dots) \\ & + (n^2(n+1)^2 - (n-1)^2 n^2) \end{aligned} \right]$$

$$= \frac{1}{4} n^2(n+1)^2$$

Using Standard Results

We have proved 3 results:—

$$\sum_{r=1}^n r^3 = \frac{1}{4} n^2 (n+1)^2$$

$$\sum_{r=1}^n r^2 = \frac{1}{6} n (n+1) (2n+1)$$

$$\sum_{r=1}^n r = \frac{1}{2} n (n+1)$$

These are also
in the formula
book

We need to be able to apply these

Example 1 Prove that

$$(1 \times 2 \times 3) + (2 \times 3 \times 4) + \dots + (n(n+1)(n+2)) = \frac{1}{4} n(n+1)(n+2)(n+3)$$

$$\sum_{r=1}^n r(r+1)(r+2) = \sum_{r=1}^n r^3 + 3r^2 + 2r$$

$$= \sum_{r=1}^n r^3 + 3 \sum_{r=1}^n r^2 + 2 \sum_{r=1}^n r$$

$$= \frac{1}{4} n^2 (n+1)^2 + \frac{1}{2} n(n+1)(2n+1) + n(n+1)$$

$$= \frac{1}{4} n(n+1) [n(n+1) + 2(2n+1) + 4]$$

$$= \frac{1}{4} n(n+1) [n^2 + n + 4n + 6]$$

$$= \frac{1}{4} n(n+1)(n+2)(n+3) \quad \underline{\text{QED}}$$

PLSD Ex 6.1 Q 3 af, 5 ad, 6, 7 ef

Example 2 Find the sum of $3^2 + 5^2 + \dots + 100^2$

$$\begin{aligned}
 \text{This is } & \sum_{r=1}^{100} r^2 - \sum_{r=1}^{49} r^2 \\
 = & \frac{1}{6} (100)(101)(201) - \frac{1}{6} (49)(50)(99) \\
 = & \underline{\underline{297925}}
 \end{aligned}$$

Example 3 Find the sum of the squares of the odd numbers from 1 to 99

$$\begin{aligned}
 \text{This is } & \sum_{r=1}^{99} r^2 - (2^2 + 4^2 + 6^2 + \dots + 98^2) \\
 = & \sum_{r=1}^{99} r^2 - \sum_{r=1}^{49} (2r)^2 \\
 = & \sum_{r=1}^{99} r^2 - 4 \sum_{r=1}^{49} r^2 \\
 = & \frac{1}{6} (99)(100)(199) - \frac{4}{6} (49)(50)(99) \\
 = & \frac{1}{6} (99)(100) [199 - 2 \times 49] \\
 = & \frac{1}{6} (99)(100)(101) \\
 = & \frac{999900}{6} = \underline{\underline{166650}}
 \end{aligned}$$