Series

The method of differences

This method of summing a series is best explained by an example.

Example 1

Find the sum of $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \dots \frac{1}{n(n+1)}$

This is $\sum_{r=1}^{n} \frac{1}{r(r+1)}$, so using partial fractions write it as

$$\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1}\right)$$

ie,

$$\left.\begin{aligned} &\left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) \\ + &\left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) \\ + &\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) \\ + &\ \cancel{-}\ \dots\dots\ \cancel{-} \\ + &\left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \end{aligned}\right\}$$

now most of the terms cancel out

$$= 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

The method works because we were able to write the general term of the series as $f(r) - f(r+1)$. This method (with variations) works on a variety of problems. A hint as to the method will normally be given.

Example 2

(a) Show that $r(r+1)(r+2) - (r-1)r(r+1) = 3r(r+1)$

(b) Hence find $\sum_{r=1}^{n} r(r+1)$

(c) Given that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$, find a formula for $\sum_{r=1}^{n} r^2$.

(a) $$r(r+1)(r+2) - (r-1)(r)(r+1) = r(r+1)\big[(r+2) - (r-1)\big]$$
$$= 3r(r+1)$$

(b) $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}\sum_{r=1}^{n}\left[r(r+1)(r+2) - (r-1)r(r+1)\right]$

$$= \frac{1}{3}\Big[(1\times\cancel{2\times3} - 0\times1\times2)$$
$$+(2\times\cancel{3\times4} - 1\times\cancel{2\times3})$$
$$+(3\times\cancel{4\times5} - 2\times\cancel{3\times4})$$
$$+(\ \cdots\ - \cdots\ )$$
$$+(n(n+1)(n+2) - \cancel{(n-1)n(n+1)})\Big]$$

$$= \underline{\underline{\frac{1}{3}n(n+1)(n+2)}}$$

(c) From (b), $\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r = \frac{1}{3}n(n+1)(n+2)$

Remember : $\sum f(r) + g(r) = \sum f(r) + \sum g(r)$

So $\sum_{r=1}^{n} r^2 = \frac{1}{3}n(n+1)(n+2) - \frac{1}{2}n(n+1)$

$$= \frac{1}{6}n(n+1)\left[2(n+2) - 3\right]$$

$$= \underline{\underline{\frac{1}{6}n(n+1)(2n+1)}}$$

<u>Example 3</u> Simplify $r^2(r+1)^2 - (r-1)^2r^2$ and hence find a formula for $\sum_{r=1}^{n} r^3$.

$$r^2(r+1)^2 - (r-1)^2r^2 = r^2\left[(r^2+2r+1) - r^2 - 2r + 1)\right]$$
$$= 4r^3$$

Hence $\sum_{r=1}^{n} r^3 = \frac{1}{4}\sum_{r=1}^{n}\left[r^2(r+1)^2 - (r-1)^2r^2\right]$

$$= \frac{1}{4}\Big[(1^2\cancel{\times 2^2} - 0^2\times1^2)$$
$$+(2^2\cancel{\times3^2} - \cancel{1^2\times} 2^2)$$
$$+(\ -\cdots\ - \ -\cdots\ )$$
$$+(n^2(n+1)^2 - \cancel{(n-1)^2n^2}\Big]$$

$$= \underline{\underline{\frac{1}{4}n^2(n+1)^2}}$$

Using Standard Results

We have proved 3 results :—

$$\boxed{\begin{aligned} \sum_{r=1}^{n} r^3 &= \frac{1}{4} n^2 (n+1)^2 \\ \sum_{r=1}^{n} r^2 &= \frac{1}{6} n(n+1)(2n+1) \\ \sum_{r=1}^{n} r &= \frac{1}{2} n(n+1) \end{aligned}}$$

These are also in the formula book

We need to be able to apply these

Example 1 Prove that

$$(1 \times 2 \times 3) + (2 \times 3 \times 4) + \cdots + (n(n+1)(n+2)) = \frac{1}{4} n(n+1)(n+2)(n+3)$$

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3r^2 + 2r$$

$$= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r$$

$$= \frac{1}{4} n^2 (n+1)^2 + \frac{1}{2} n(n+1)(2n+1) + n(n+1)$$

$$= \frac{1}{4} n(n+1) \left[n(n+1) + 2(2n+1) + 4 \right]$$

$$= \frac{1}{4} n(n+1) \left[n^2 + n + 4n + 6 \right]$$

$$= \frac{1}{4} n(n+1)(n+2)(n+3)$$

QED

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Example 2 Find the sum of $50^2 + 51^2 + \cdots\cdots 100^2$

This is $\sum_{r=1}^{100} r^2 - \sum_{r=1}^{49} r^2$

$$= \frac{1}{6}(100)(101)(201) - \frac{1}{6}(49)(50)(99)$$

$$= 297925$$

Example 3 Find the sum of the squares of the odd numbers from 1 to 99

This is $\sum_{r=1}^{99} r^2 - (2^2 + 4^2 + 6^2 + \cdots 98^2)$

$$= \sum_{r=1}^{99} r^2 - \sum_{r=1}^{49} (2r)^2$$

$$= \sum_{r=1}^{99} r^2 - 4\sum_{r=1}^{49} r^2$$

$$= \frac{1}{6}(99)(100)(199) - \frac{4}{6}(49)(50)(99)$$

$$= \frac{1}{6}(99)(100)\left[199 - 2 \times 49\right]$$

$$= \frac{1}{6}(99)(100)(101)$$

$$= \frac{999900}{6} = 166650$$