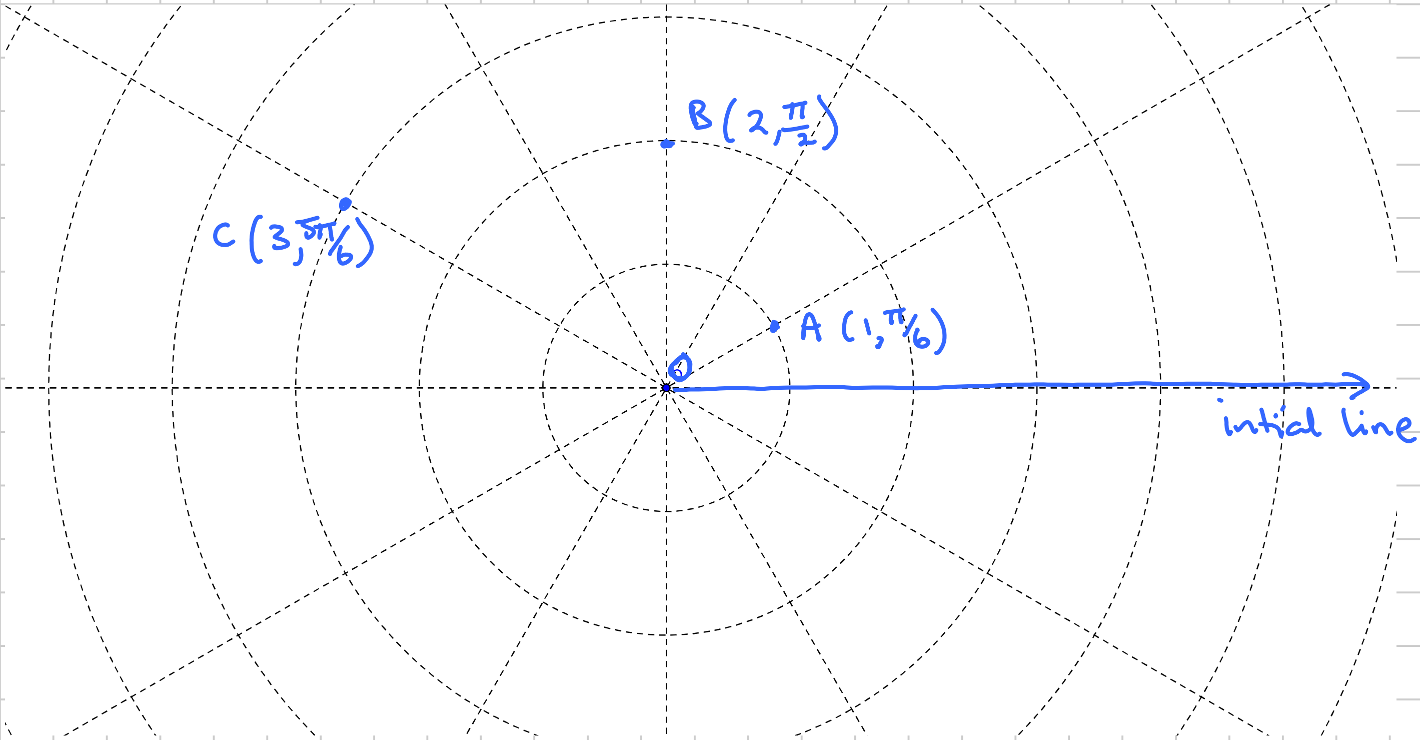
Polar Coordinates

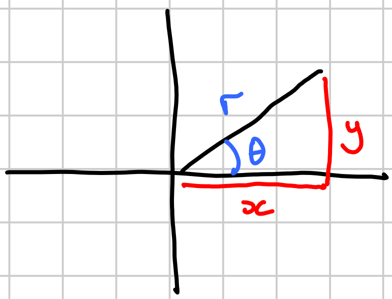
These are an alternative to the usual Cartesian coordinates (x, y), in which a point is defined as $P(r, \theta)$, where:

r = distance to P from a point O, the 'Pole' or origin
θ = angle which OP makes with the 'Initial line', a line pointing in the positive x-direction



For Edexcel, r is restricted to positive values, and $-\pi < \theta \leq \pi$, so that the coordinates of a point are unique (otherwise C could be $(-3, -\frac{\pi}{6})$ or $(-3, \frac{11\pi}{6})$ etc).
But some boards, books etc allow r to be negative.

We can switch between polar and cartesian coords :-

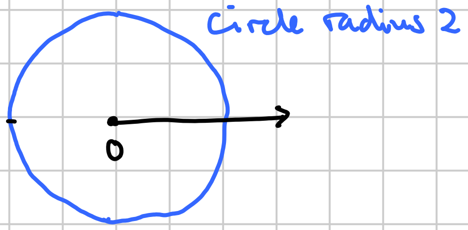


$$\boxed{\begin{array}{ll} r = \sqrt{x^2+y^2} & x = r\cos\theta \\ \theta = \arctan\left(\frac{y}{x}\right) & y = r\sin\theta \end{array}}$$

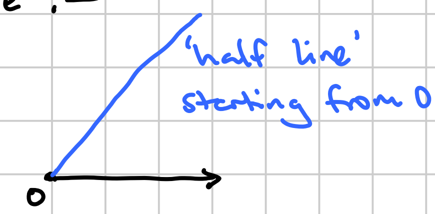
So the Cartesian coords of C above are $\left(3\cos\frac{5\pi}{6}, 3\sin\frac{5\pi}{6}\right)$
ie, $\left(\frac{-3\sqrt{3}}{2}, \frac{3}{2}\right)$

Polar graphs

Some simple polar equation are easy to visualise :–

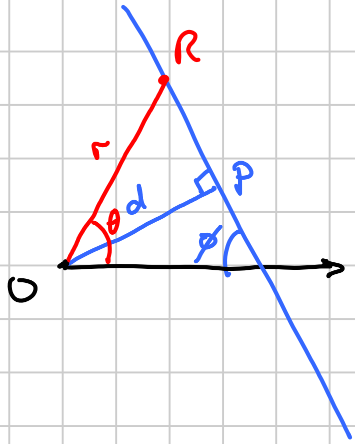


$r = 2$



$\theta = \frac{\pi}{4}$

To find the equation of a general straight line, which makes an angle ϕ with the initial line, and has a shortest distance to the pole O of d :–



Consider a general point R on the line :

$$R\hat{O}P = \theta - \left(\frac{\pi}{2} - \phi\right)$$

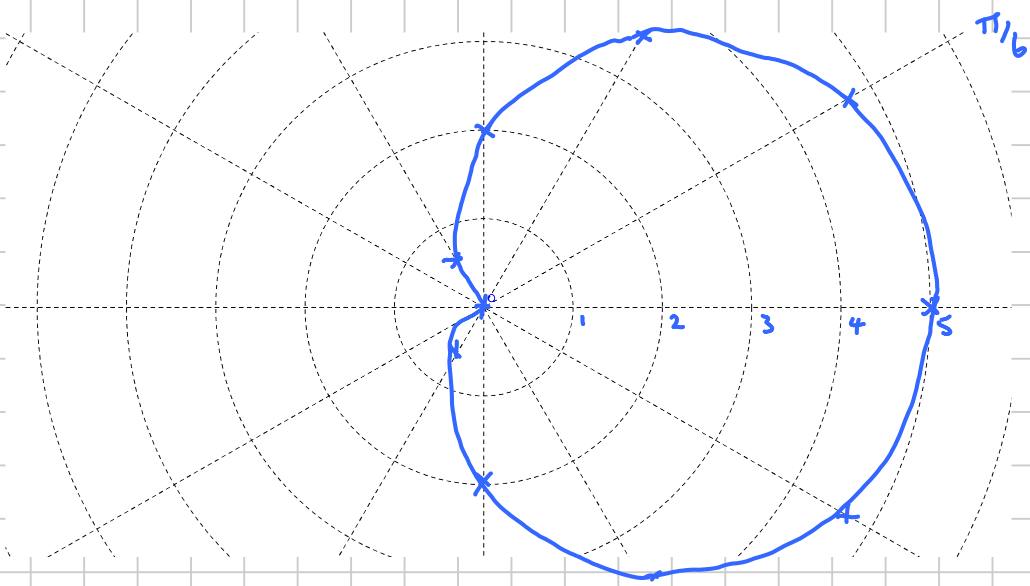
So
$$\frac{d}{r} = \cos\left(\theta + \phi - \frac{\pi}{2}\right)$$
$$= \sin(\theta + \phi)$$

$$\underline{r = d \operatorname{cosec}(\theta + \phi)}$$

Example By drawing up a table of values, plot the graph of $r = 2 + 3\cos\theta$

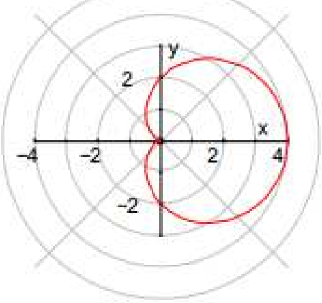
θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	2.3	$5\pi/6$	π
r	5	4.6	3.5	2	0.5	0	(–ve)	(–ve)

(since $\cos(-\theta) = \cos\theta$ the values for $\theta = -\pi/6$ etc will be the same)

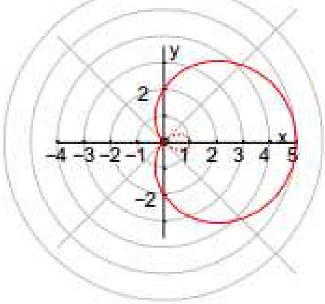


More examples :

Polar Graphs



$r = 2(1 + \cos\theta)$



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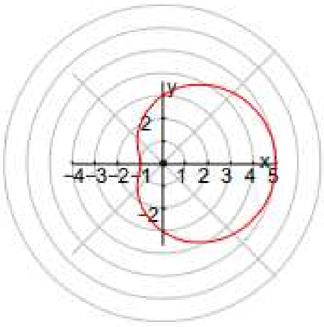
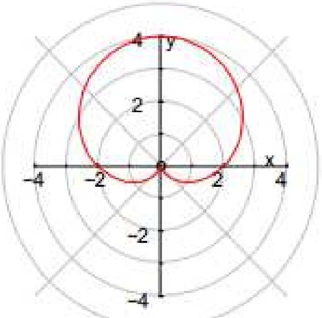
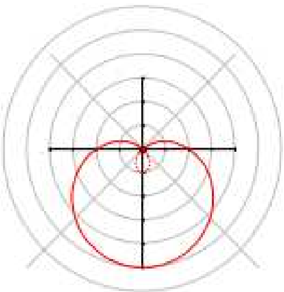


$r = 3 + 2\cos\theta$

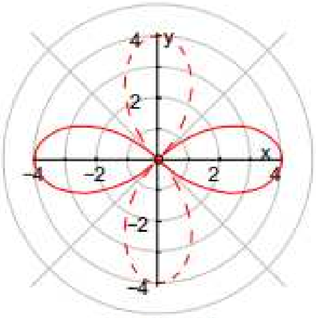


$r = 2(1 + \sin\theta)$

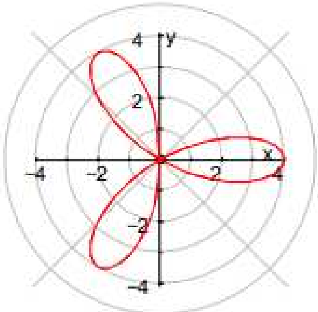


$r = 2 - 3\sin\theta$

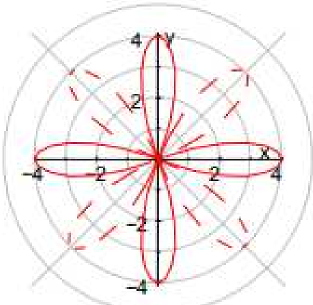
(Note that dotted parts of the graphs represent points where r is negative, so should be ignored!)



$r = 4\cos2\theta$



$r = 4\cos3\theta$



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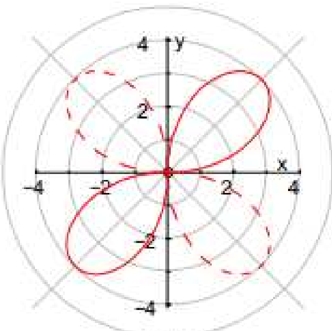

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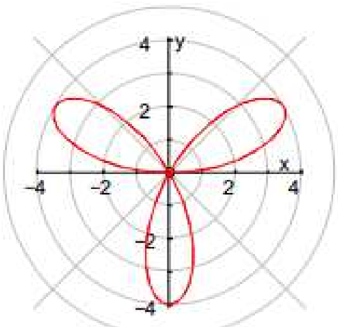

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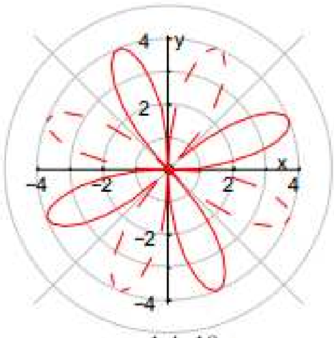

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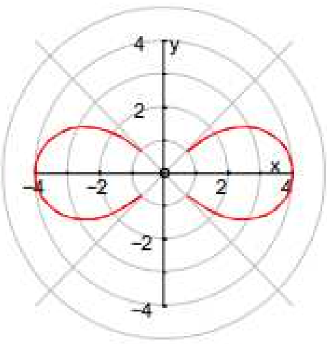


$r^2 = 16\cos2\theta$

Converting between polar and Cartesian equations

Examples

① Write $r = 2 + 3\cos\theta$ in Cartesian form

Use $r = \sqrt{x^2+y^2}$ and $r\cos\theta = x$

$$\sqrt{x^2+y^2} = 2 + 3\frac{x}{r}$$

$$= 2 + \frac{3x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 2\sqrt{x^2+y^2} + 3x$$

$$x^2+y^2-3x = 2\sqrt{x^2+y^2}$$

$$\underline{(x^2+y^2-3x)^2 = 4(x^2+y^2)}$$

② $x^2 - 2ax + y^2 = 0$ is a circle centre $(a, 0)$, radius a.

Write this in polar form

$x^2+y^2 = r^2$, $x = r\cos\theta$

$$r^2 - 2ar\cos\theta = 0$$

$$r(r - 2a\cos\theta) = 0$$

Either $r = 0$ (which is a single point, and is included in the equation below and so can be ignored)

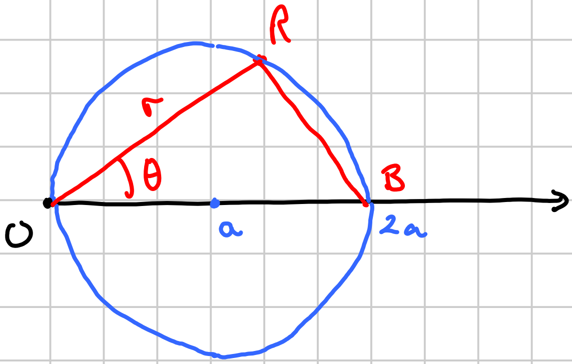
or $\underline{r = 2a\cos\theta}$

(Another way to see this is:

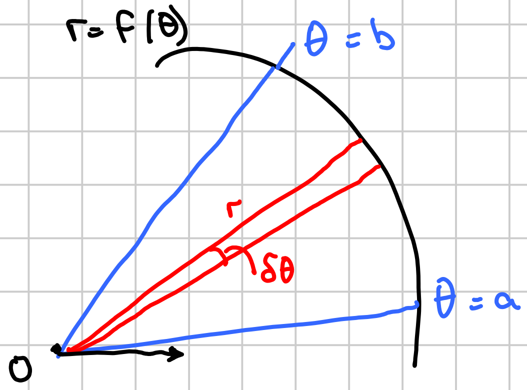
For any R on the circle, $O\hat{R}B$ is a right angle (angle in semicircle)

so $\cos\theta = \frac{r}{2a}$

$r = 2a\cos\theta$)



Area enclosed by a Polar curve



Suppose we want to find the area enclosed by a curve $r = f(\theta)$ and two half-lines $\theta = a$ and $\theta = b$

We can divide the area into small elements with angle $\delta\theta$ (shown in red). Each element is approximately a sector of a circle, so its area $\approx \frac{1}{2}r^2\delta\theta$.

So the total area is the sum of these elements:–

$$A \approx \sum_{\theta=a}^{b} \frac{1}{2}r^2\delta\theta$$

By letting $\delta\theta \to 0$, the approximation becomes exact:–

$$A = \lim_{\delta\theta\to 0} \sum_{\theta=a}^{b} \frac{1}{2}r^2\delta\theta$$

As usual, $\lim \sum$ is an integral:–

$$\boxed{A = \int_{\theta=a}^{b} \frac{1}{2}r^2\,d\theta}$$

Examples

① Find the area enclosed between the circle $r = 2a\cos\theta$ and the half-lines $\theta = -\pi/4$ and $\theta = \pi/4$

$$\text{Area} = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(2a\cos\theta)^2\,d\theta$$

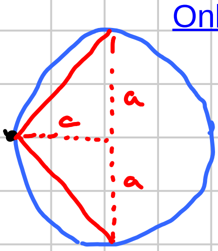
$$= 2a^2\int_{-\pi/4}^{\pi/4} \cos^2\theta\,d\theta$$

(Use $\cos 2\theta = 2\cos^2\theta - 1$)

$$= 2a^2\int_{-\pi/4}^{\pi/4} \frac{1}{2}(1 + \cos 2\theta)\,d\theta$$

$$= a^2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4}$$

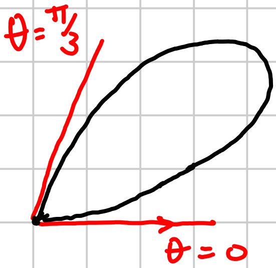
$$= a^2\left[\left(\frac{\pi}{4} + \frac{1}{2}\right) - \left(-\frac{\pi}{4} - \frac{1}{2}\right)\right] = \underline{a^2 + \frac{1}{2}\pi a^2}$$

[Check: 

Area is a semicircle radius a + a triangle

$= \frac{1}{2}\pi a^2 + \frac{1}{2} \times 2a \times a$]

If the curve completely encloses the area, meeting at the pole, we need to find the values of θ for which $r = 0$ (these half-lines are tangent to the curve)

② 

The curve shows one loop of the graph of $r = \sin 3\theta$. Find the area enclosed.

First let $r = 0 \Rightarrow \sin 3\theta = 0$

$3\theta = 0, \pi, 2\pi \ldots$

$\theta = 0, \pi/3, 2\pi/3 \ldots$

$$\text{So area} = \int_{\theta=0}^{\pi/3} \tfrac{1}{2}\sin^2 3\theta \, d\theta$$

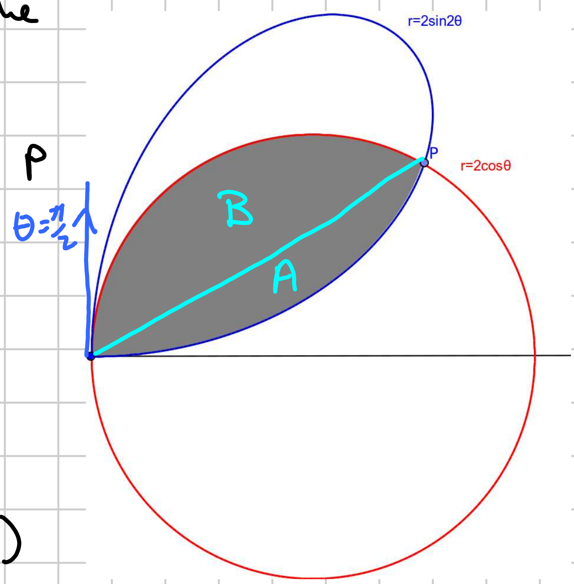
(Use $\cos 2A = 1 - 2\sin^2 A$)
$$= \tfrac{1}{2}\int_0^{\pi/3} \tfrac{1}{2}(1 - \cos 6\theta)\, d\theta$$

$$= \tfrac{1}{4}\left[\theta - \tfrac{1}{6}\sin 6\theta\right]_0^{\pi/3}$$

$$= \tfrac{1}{4}\left[\left(\tfrac{\pi}{3} - 0\right) - (0 - 0)\right] = \underline{\underline{\tfrac{\pi}{12}}}$$

We can find the area between two curves using addition or subtraction

③ Find the shaded area in the diagram on the right



First we find the value of θ at P

$\cancel{2}\sin 2\theta = \cancel{2}\cos\theta$

$2\sin\theta\cos\theta = \cos\theta$

$\cos\theta(2\sin\theta - 1) = 0$

$\cos\theta = 0$ (at the pole)

or $\sin\theta = \frac{1}{2} \Rightarrow \underline{\theta = \pi/6}$ (at P)

So shaded area = Area A + Area B

$$= \frac{1}{2}\int_0^{\pi/6} (2\sin 2\theta)^2 \, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (2\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/6} 4\times\tfrac{1}{2}(1-\cos 4\theta) + \frac{1}{2}\int_{\pi/6}^{\pi/2} 4\times\tfrac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/6} + \left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/2}$$

$$= \left[\left(\pi/6 - \frac{\sqrt{3}}{8}\right) - (0-0)\right] + \left[\left(\pi/2 + 0\right) - \left(\pi/6 + \frac{\sqrt{3}}{4}\right)\right]$$

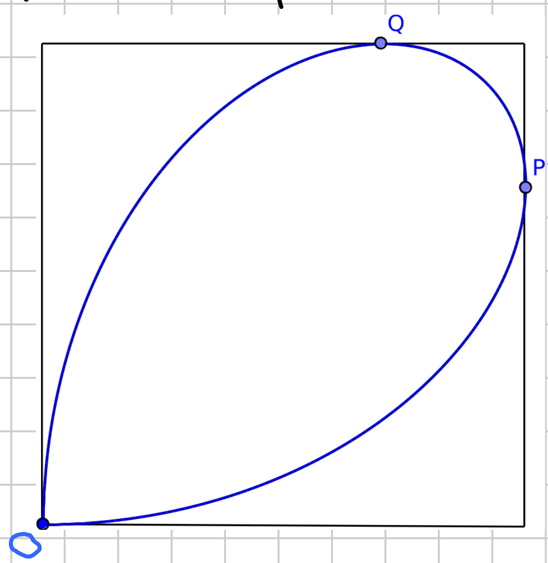
$$= \underline{\frac{\pi}{2} - \frac{3\sqrt{3}}{8}}$$

Turning points

By putting $\frac{dr}{d\theta}$ equal to zero, we can find where r is a maximum or minimum ie, the point(s) on the curve which are furthest from or closest to the pole.

But more often we want to find where the tangent to the curve is horizontal (ie where y has a turning point) or vertical (where x has a turning point). To do this we need to put $\frac{dy}{d\theta}$ or $\frac{dx}{d\theta}$ equal to 0.

Example The diagram shows a loop of the curve $r = 3\sin 2\theta$, enclosed in a rectangle. Find the polar coords of P and Q, and show that the area of the rectangle is $\frac{16}{3}$.



At Q, $\frac{dy}{d\theta} = 0$, so $\frac{d}{d\theta}(r\sin\theta) = 0$

$$\Rightarrow \frac{d}{d\theta}(3\sin 2\theta \sin\theta) = 0$$

$$6\cos 2\theta \sin\theta + 3\sin 2\theta \cos\theta = 0$$

$$6(2\cos^2\theta - 1)\sin\theta + 6\cos^2\theta \sin\theta = 0$$

$$6\sin\theta(2\cos^2\theta - 1 + \cos^2\theta) = 0$$

either $\sin\theta = 0 \Rightarrow \theta = 0$ (at O)

or $3\cos^2\theta = 1$

$\cos\theta = \frac{1}{\sqrt{3}}$ (at Q)

So at Q, $\theta = \arccos\left(\frac{1}{\sqrt{3}}\right)$ and $r = 3 \times 2\sin\theta\cos\theta$

$= 6 \times \frac{1}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}}$

$= 2\sqrt{2}$

$\left[\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1 - 1/3} = \sqrt{2/3} \right]$

Now at P, $\frac{dx}{d\theta} = 0$, so $\frac{d}{d\theta}(r\cos\theta) = 0$

$\Rightarrow \frac{d}{d\theta}(3\sin 2\theta\cos\theta) = 0$

$\Rightarrow 6\cos 2\theta\cos\theta - 3\sin 2\theta\sin\theta = 0$

$\Rightarrow 6(2\cos^2\theta - 1)\cos\theta - 6\cos\theta\sin^2\theta = 0$

$\Rightarrow 6\cos\theta\left(2\cos^2\theta - 1 - (1-\cos^2\theta)\right) = 0$

So either $6\cos\theta = 0 \Rightarrow \theta = \pi/2$ (at O)

or $3\cos^2\theta = 2 \Rightarrow \cos\theta = \sqrt{\frac{2}{3}}$ (at P)

So at P, $\theta = \arccos\left(\sqrt{\frac{2}{3}}\right)$ and $r = 3 \times 2\sin\theta\cos\theta$

$= 3 \times 2\frac{1}{\sqrt{3}} \times \frac{\sqrt{2}}{\sqrt{3}}$

$= 2\sqrt{2}$

To find the area: at Q, $y = r\sin\theta$

$= 2\sqrt{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{4}{\sqrt{3}}$

at P, $x = r\cos\theta$

$= 2\sqrt{2} \times \frac{\sqrt{2}}{\sqrt{3}} = \frac{4}{\sqrt{3}}$

So the rectangle is actually a square, with area $\frac{16}{3}$