Matrices

Transformations

A matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be used to represent a linear transformation of the x-y plane to the x'-y' plane.

ie/ $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

where $\underline{r} = x\underline{i} + y\underline{j}$ is the pv of a point R and $\underline{r}' = x'\underline{i} + y'\underline{j}$ is the pv of R', the image of R.

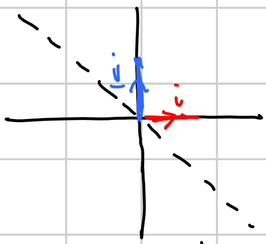
Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the origin is an <u>invariant point</u> under any matrix transformation.

The point with p.v. $\underline{i}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \underline{p}$

The point with pv $\underline{j}$ is mapped to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} = \underline{q}$

This gives a way of finding the matrix for a given transformation.

e.g. a reflection in the line $y = -x$



$$\underline{i} \to \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

and $$\underline{j} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

so the matrix for this transformation is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

We can view the transformation in two ways :-

<u>either</u> $x\underline{i} + y\underline{j} \longrightarrow x'\underline{i} + y'\underline{j}$ where $x' = ax + by$, $y' = cx + dy$

(the base vectors stay the same, but the components change) or $\quad x\underline{i} + y\underline{j} \to x\underline{p} + y\underline{q}$ $\quad$ ie, $x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$

(the components / coefficients stay the same, but the base vectors change)

eg if $M = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$ then a point A with pv $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

is transformed to $\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\underline{i} + 5\underline{j}$

or is transformed to $2\underline{p} + 1\underline{q}$ $\left[= 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} \right]$

[If we view the plane as a grid of unit squares, with sides $\underline{i}$ and $\underline{j}$, it is transformed to a grid of parallelograms with sides $\underline{p}$ and $\underline{q}$.]

Singular Matrices

If $\underline{q}$ is a multiple of $\underline{p}$ (eg $M = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$ so that $\underline{q} = -\frac{1}{2}\underline{p}$) then the whole plane is mapped to the line $r = \lambda\underline{p}$. The transformation is MANY-TO-ONE ie, many points in the plane (a whole line of them) are mapped to a single point on the image line. In particular, a whole line of points is mapped to the origin

e.g. $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 1\lambda \\ 2\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad$ so under this transformation the line $r = \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is mapped to the origin.

In this case A is called SINGULAR matrix

Orthogonal Matrices

If $\underline{p}$ and $\underline{q}$ are perpendicular unit vectors (ie, $|\underline{p}| = |\underline{q}| = 1$ and $\underline{p}.\underline{q} = 0$) then M represents a rigid transformation (an isometry) of the plane. In this case M is called an orthogonal matrix. In 2D, orthogonal matrices represent rotations or reflections.

Some common transformations

Enlargement by scale factor k $\qquad \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

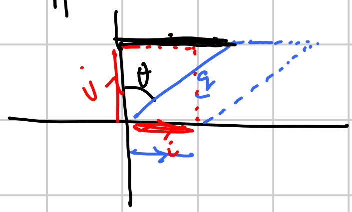
Stretch in x-direction by scale factor k $\qquad \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

Rotation through θ about the origin
(+ve angle $\Rightarrow$ anticlockwise) $\qquad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Reflection in the line $y = mx$
where $m = \tan\theta$ $\qquad \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Shear $\parallel$ to x-axis by angle θ $\qquad \begin{pmatrix} 1 & \tan\theta \\ 0 & 1 \end{pmatrix}$

$\cos\theta = \dfrac{1}{|q|}$



Compound Transformations

If transformation P is represented by matrix M and Q by matrix N, then the transformation "P followed by Q" is calculated using

$$N\left(M\begin{pmatrix} x \\ y \end{pmatrix}\right) \quad \text{or} \quad NM\begin{pmatrix} x \\ y \end{pmatrix}$$

ie the single matrix for "P followed by Q" is NM
(note the order!)

Determinant and Area

The DETERMINANT of a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|M| = ad - bc$. The transformation M multiplies the area of any shape in the plane by $|M|$. If $|M|$ is negative, the shape is 'turned over' (as in a reflection) by the transformation.

e.g. for a straightforward reflection

$$\begin{vmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{vmatrix} = -\cos^2 2\theta - \sin^2 2\theta = -1$$

If $|M| = 0$, area is destroyed, so M is a SINGULAR matrix which maps the plane to a line (or possibly to a point if $M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$)

Inverse Matrices and Transformations

The inverse of a matrix is written M^{-1} and is defined by

$$M^{-1}M = I$$ where I is the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\text{The inverse of } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{ie, } \frac{1}{|M|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

e.g. if $M = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, $|M| = 10$

so $M^{-1} = \frac{1}{10}\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$ or $\begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}$

Note that:

1. If $|M| = k$, $|M^{-1}| = \frac{1}{k}$ $\quad (|M^{-1}| = |M|^{-1})$
2. If $|M| = 0$ M has no inverse (because M is a many-to-one transformation)

③ $(MN)^{-1} = N^{-1}M^{-1}$ (undo transformations in reverse order).

The transpose of a matrix

If $M = \begin{array}{c} \quad \underline{p} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$, $M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

(rows $\longleftrightarrow$ columns, or reflect in leading diagonal)

Note that :-

① If M is orthogonal

$|\underline{p}| = |\underline{q}| = 1$ so $\sqrt{a^2+c^2} = 1 = \sqrt{b^2+d^2}$

also $\underline{p} \cdot \underline{q} = 0$ so $ab + cd = 0$.

Now consider $M^TM = \begin{pmatrix} \boxed{a \quad c} \\ b \quad d \end{pmatrix}\begin{pmatrix} \boxed{\begin{matrix} a \\ c \end{matrix}} & \begin{matrix} b \\ d \end{matrix} \end{pmatrix}$

$= \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

So for an orthogonal matrix M, $M^{-1} = M^T$

② For any square matrices A and B

$$(AB)^T = B^TA^T$$

③ If $M^T = M$, M is called a SYMMETRIC matrix

e.g $\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$

3D and 3×3 Matrices

All the above extends in a natural way to 3D space (and indeed to n-D space)

$\underline{i}, \underline{j}, \underline{k}$ are transformed to $\underline{p}, \underline{q}, \underline{r}$, the columns of M and so a unit cube becomes a parallelepiped.

Singular 3×3 Matrix

If $|M| = 0$, M is singular so 3D space is transformed to :-

- a point (if M is the zero matrix)
- a line (if $\underline{p}$, $\underline{q}$ and $\underline{r}$ are all multiples of each other)
- a plane (if $\underline{p}$, $\underline{q}$ and $\underline{r}$ all lie in the same plane)

3×3 Determinants

Let $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

If we choose any element and draw a horizontal and vertical line through it we are left with a 2×2 matrix.

The determinant of this is called the MINOR of the element. So the MINOR of 3 is $4\times 8 - 5\times 7 = -3$.

To take account of the 'wrapping round' left to right and top to bottom we attach a sign to this minor according to the following scheme

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

This gives a number called the COFACTOR of the element. So the cofactor of 3 is still -3

But for 2, the MINOR is $4\times 9 - 6\times 7 = -6$

but the COFACTOR is $-(-6) = 6$

Now the determinant of M is found by choosing any row or column of M, multiplying each element in the row column by its cofactor and adding the results.

So $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} =$ (using the top row)

$1 \times (5\times 9 - 6 \times 8) + 2 \times 6 + 3 \times -3$

$= -3 + 12 - 9$

$= 0$

e.g. $\begin{vmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ 0 & 5 & 2 \end{vmatrix}$ $\quad 1 \times 6 - 5$

(use the bottom row) $= 0 + 5 \times -(1 - -8) + 2 \times (3 - -4)$

$= -45 + 14$

$= -31$

Inverse of a 3×3 matrix

Example Let $M = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}$

① Find the determinant : $|M| = 4 \times -(1 - -6) + 1 \times (-3 - -4)$

$= -28 + 1 = -27$

② Find M^T $\quad M^T = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -3 & 4 \\ 3 & 1 & 1 \end{pmatrix}$

③ Replace each element of M^T by its COFACTOR. The resulting matrix is called the ADJOINT of M

$$\operatorname{adj}(M) = \begin{pmatrix} -7 & 10 & 11 \\ 2 & 1 & -7 \\ -8 & -4 & 1 \end{pmatrix}$$

④ Divide adj(M) by |M| to get M^{-1}

$$M^{-1} = \frac{1}{27}\begin{pmatrix} +7 & -10 & -11 \\ -2 & -1 & +7 \\ +8 & 4 & -1 \end{pmatrix}$$

⑤ Check!

$$M^{-1}M = \frac{1}{27}\begin{pmatrix} 7 & -10 & -11 \\ -2 & -1 & 7 \\ 8 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

$$= \frac{1}{27}\begin{pmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix} = I$$

Image of a line and a plane

Example Find the image of

(a) the line $\frac{x-1}{2} = \frac{z-1}{3}$; $y=2$

(b) the plane $x - 2y - z = 2$

under the transformation $M = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}$

(a) this is best handled with the line in vector form

$$\frac{x-1}{2} = \frac{z-1}{3} = \lambda \Rightarrow \begin{aligned} x &= 1 + 2\lambda \\ y &= 2 \\ z &= 1 + 3\lambda \end{aligned}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$$\underline{r} = \underline{a} + \lambda \underline{v}$$

Hence $$M\underline{r} = M\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 1 \\ 0 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

(b) This is best tackled using the inverse matrix.

Let the new coordinate system be $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{27} \begin{pmatrix} 7 & -10 & -11 \\ -2 & -1 & 7 \\ 8 & 4 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

$$x = \tfrac{1}{27}(7x' - 10y' - 11z')$$
$$y = \tfrac{1}{27}(-2x' - 1y' + 7z')$$
$$z = \tfrac{1}{27}(8x' + 4y' - z')$$

But we want to consider points on the plane

$$x - 2y - z = 2$$

$$\frac{1}{27}\Big(7x' - 10y' - 11z' - 2(-2x' - y' + 7z') - (8x' + 4y' - z')\Big) = 2$$

$$3x' - 12y' - 24z' = 54$$

$$x' - 4y' - 8z' = 18 \qquad \left[\text{or } \mathbf{r}'.(\underline{i} - 4\underline{j} - 8\underline{k}) = 18\right]$$

p74 Ex 3B Q 4 (some until confident)

p165 Review Ex. Q 3, 9, 15, 27, 29, 41, 44, 69

Eigenvalues and Eigenvectors

Consider the product

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad \text{which is just } 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

ie, the effect of multiplying $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ by the matrix $M = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ is the same as multiplying by the scalar 2.

We say that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an EIGENVECTOR of M, with EIGENVALUE 2.

Eigenvectors are not unique – any vector $k\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is also an eigenvector of M with eigenvalue 2.

So the line $r = \lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ (through the origin)

is mapped to $r' = Mr$

ie, $r' = 2\lambda\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

which is the same line 'stretched' by a factor 2.

We say that $r = \lambda\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is an INVARIANT LINE under the transformation M.

If the eigenvalue were 1, ie, $M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ for some vector $\begin{pmatrix} x \\ y \end{pmatrix}$, the line is called a LINE OF INVARIANT POINTS.

To find the evalues and evectors of a given matrix, we use the following theory :–

Given M, we need λ and $\underline{v}$ such that

$$M\underline{v} = \lambda \underline{v}$$
$$\Rightarrow \quad M\underline{v} - \lambda \underline{v} = \underline{0} \text{ or } \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\Rightarrow \quad M\underline{v} - \lambda I \underline{v} = \underline{0}$$
$$\Rightarrow \quad (M - \lambda I)\underline{v} = \underline{0}$$

This statement says that the matrix $M - \lambda I$ transforms the non-zero vector $\underline{v}$ to the origin $\underline{0}$.
This is only possible is $M - \lambda I$ is a SINGULAR matrix ie if

$$\boxed{|M - \lambda I| = 0}$$

This is called the 'characteristic equation' of M and is the starting point for finding the eigenvalues.

<u>Example 1</u>

(a) Find the evalues of $M = \frac{1}{10}\begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}$ and find a normalized (ie unit) evector associated with each evalue

(b) Hence describe geometrically the transformation represented by this matrix.

(a) $$|M - \lambda I| = 0$$

$$\left| \begin{pmatrix} \frac{11}{10} & \frac{3}{10} \\ \frac{3}{10} & \frac{19}{10} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} \frac{11}{10} - \lambda & \frac{3}{10} \\ \frac{3}{10} & \frac{19}{10} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{11}{10} - \lambda\right)\left(\frac{19}{10} - \lambda\right) - \frac{9}{100} = 0$$

$$\frac{209}{100} - \frac{30}{10}\lambda + \lambda^2 - \frac{9}{100} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 2)(\lambda - 1) = 0$$

$$\lambda = 2 \quad \text{or} \quad \lambda = 1$$

Eigenvectors for $\lambda = 2$:

$$\frac{1}{10}\begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} 11x + 3y &= 20x \\ 3x + 19y &= 20y \end{aligned}\right\}$$

$$\left.\begin{aligned} -9x + 3y &= 0 \\ 3x - y &= 0 \end{aligned}\right\}$$ these are the same equation, as we would expect

Let $x = k_1 \Rightarrow y = 3k_1$

So the eigenvectors are $\begin{pmatrix} k_1 \\ 3k_1 \end{pmatrix}$ or $k_1\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The unit eigenvector is $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ $\left(= \underline{b}\right)$

[The vector eqn of the invariant line is $\underline{r} = \lambda\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

The Cartesian eqn is $y = 3x$]

For $\lambda = 1$:

$$\frac{1}{10}\begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\left.\begin{aligned} 11x + 3y &= 10x \\ 3x + 19y &= 10y \end{aligned}\right\}$$

$$\left.\begin{aligned} x + 3y &= 0 \\ 3x + 9y &= 0 \end{aligned}\right\}$$

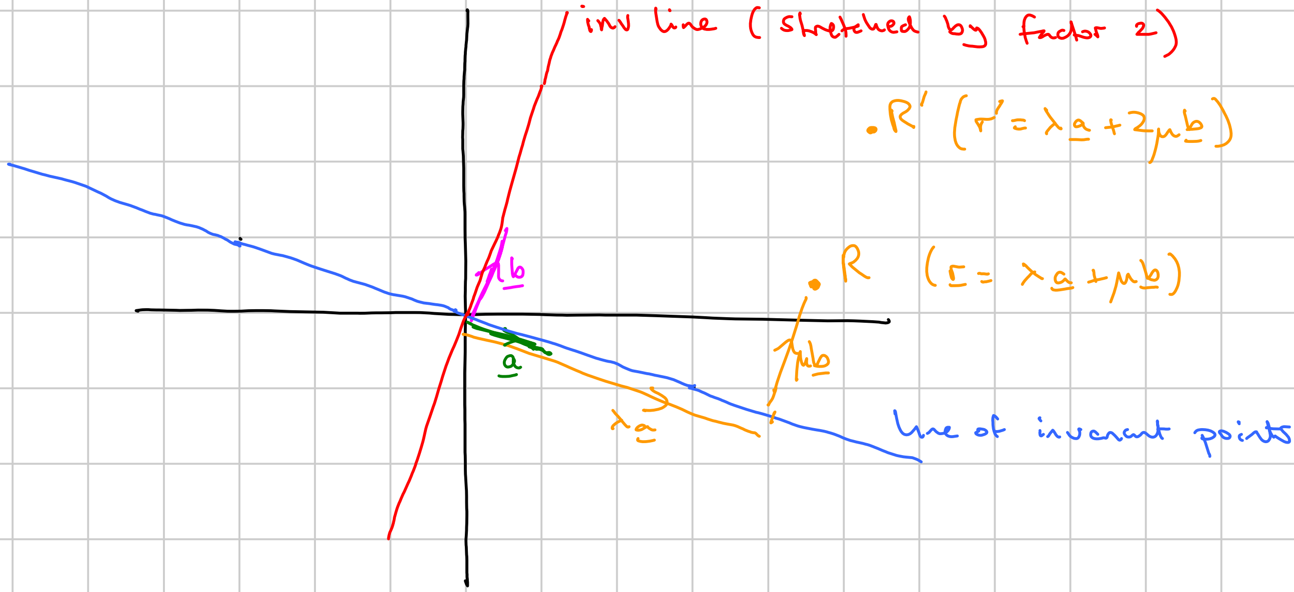
Let $y = k_2 \Rightarrow x = -3k_2$

Eigenvectors are $k_2\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Unit eigenvector is $\frac{1}{\sqrt{10}}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ Let $\underline{a} = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

So $y = -\frac{1}{3}x$ is a line of invariant points

(b)



The pv of any point R in the plane can be written as

$$\underline{r} = \lambda\underline{a} + \mu\underline{b}$$

This is transformed by M to

$$\begin{aligned} \underline{r}' &= M\underline{r} \\ &= M(\lambda\underline{a} + \mu\underline{b}) \\ &= \lambda M\underline{a} + \mu M\underline{b} \\ &= \lambda\underline{a} + 2\mu\underline{b} \end{aligned}$$

The transformation is a stretch by scale factor 2 perpendicular to the line of invariant points $y = -\frac{1}{3}x$.

Example 2 Given that $\lambda = 2$ is an eigenvalue of $M = \frac{1}{3}\begin{pmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix}$, find all the evalues, and find a set of 3 mutually perpendicular evectors for M.

$$|M - \lambda I| = 0$$

$$\begin{vmatrix} 5/3 - \lambda & -2/3 & 2/3 \\ -2/3 & 2/3 - \lambda & 4/3 \\ 2/3 & 4/3 & 2/3 - \lambda \end{vmatrix} = 0$$

$$\left(\tfrac{5}{3}-\lambda\right)\left[\left(\tfrac{2}{3}-\lambda\right)\left(\tfrac{2}{3}-\lambda\right)-\tfrac{16}{9}\right]+\tfrac{2}{3}\left[-\tfrac{2}{3}\left(\tfrac{2}{3}-\lambda\right)-\tfrac{8}{9}\right]+\tfrac{2}{3}\left[-\tfrac{8}{9}-\tfrac{2}{3}\left(\tfrac{2}{3}-\lambda\right)\right]=0$$

$$\left(\tfrac{5}{3}-\lambda\right)\left[\tfrac{4}{9}-\tfrac{4}{3}\lambda+\lambda^2-\tfrac{16}{9}\right]-\tfrac{8}{27}+\tfrac{4}{9}\lambda-\tfrac{16}{27}-\tfrac{16}{27}-\tfrac{8}{27}+\tfrac{4}{9}\lambda=0$$

$$-\frac{60}{27}+\frac{12}{9}\lambda-\frac{20}{9}\lambda+\frac{4}{3}\lambda^2+\frac{5}{3}\lambda^2-\lambda^3-\frac{48}{27}+\frac{8}{9}\lambda=0$$

$$-\frac{108}{27}+\frac{9}{3}\lambda^2-\lambda^3=0$$

$$\lambda^3-3\lambda^2+4=0$$

$\lambda=2$ is a root $\Rightarrow$ $\lambda-2$ is a factor

$$\begin{array}{r|l} & \lambda^2 \quad -\lambda \quad -2 \\ \hline \lambda-2 & \lambda^3-3\lambda^2+0\lambda+4 \\ & \underline{\lambda^3-2\lambda^2} \\ & \quad -\lambda^2+0\lambda \\ & \quad \underline{-\lambda^2+2\lambda} \\ & \qquad -2\lambda+4 \\ & \qquad \underline{-2\lambda+4} \\ & \qquad\qquad 0 \end{array}$$

$$\lambda^3-3\lambda^2+4=0$$

$$(\lambda-2)(\lambda-2)(\lambda+1)=0$$

Eigenvalues are $\lambda=2$ (repeated)

$\lambda=-1$

Eigenvectors for $\lambda=-1$

$$\frac{1}{3}\begin{pmatrix}5 & -2 & 2\\ -2 & 2 & 4\\ 2 & 4 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\left.\begin{aligned}5x-2y+2z&=-3x\\ -2x+2y+4z&=-3y\\ 2x+4y+2z&=-3z\end{aligned}\right\}$$

$$\left.\begin{aligned} 8x - 2y + 2z &= 0 \quad (1)\\ -2x + 5y + 4z &= 0 \quad (2)\\ 2x + 4y + 5z &= 0 \quad (3)\end{aligned}\right\}$$

Eliminate z:

$2\times(1) - (2) \Rightarrow 18x - 9y = 0$

$\Rightarrow 2x - y = 0$

Let $x = k_1 \Rightarrow y = 2k_1$

Subst into (1) $\Rightarrow 8k_1 - 4k_1 + 2z = 0$

$z = -2k_1$

Eigenvectors are $k_1\begin{pmatrix}1\\2\\-2\end{pmatrix}$

Unit eigenvector is $\frac{1}{3}\begin{pmatrix}1\\2\\-2\end{pmatrix}$

Eigenvectors for $\lambda = 2$:

$$\frac{1}{3}\begin{pmatrix}5 & -2 & 2\\ -2 & 2 & 4\\ 2 & 4 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\left.\begin{aligned} 5x - 2y + 2z &= 6x\\ -2x + 2y + 4z &= 6y\\ 2x + 4y + 2z &= 6z\end{aligned}\right\}$$

$$\left.\begin{aligned} -x - 2y + 2z &= 0\\ -2x - 4y + 4z &= 0\\ 2x + 4y - 4z &= 0\end{aligned}\right\}$$

These are all the same equation, which is therefore the equation of a plane, which is stretched by scale factor 2 by M

We need two parameters:

Let $y = k_2$ and $z = k_3$ so $x = -2k_2 + 2k_3$

Eigenvectors for $\lambda = 2$ are of the form $\begin{pmatrix} -2k_2 + 2k_3 \\ k_2 \\ k_3 \end{pmatrix}$

or $k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

[So a parametric vector eqn of this plane is

$$\underline{r} = \underline{0} + k_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + k_3 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$]

Now

$$k_1 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -2k_2 + 2k_3 \\ k_2 \\ k_3 \end{pmatrix} = k_1 \left[(-2k_2 + 2k_3) + 2k_2 - 2k_3 \right]$$

$$= 0$$

So the eigenvectors for $\lambda = -1$ are perpendicular to the plane of eigenvectors for $\lambda = 2$.

So we need 2 vectors in the plane perpendicular to each other.

Choose $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ as one of them.

Then we need $\begin{pmatrix} -2k_2 + 2k_3 \\ k_2 \\ k_3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = 0$

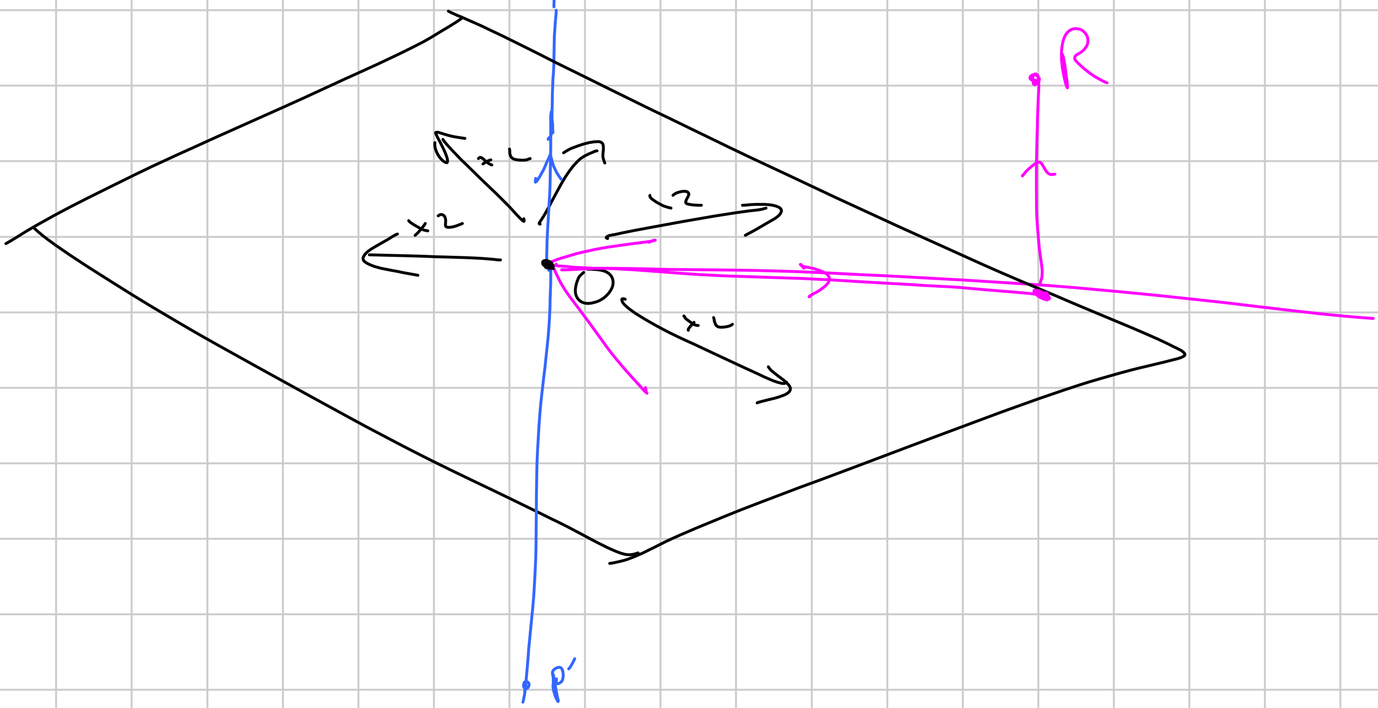
$$-4k_2 + 4k_3 + k_3 = 0$$

$$-4k_2 + 5k_3 = 0$$

e.g. $k_2 = 5$, $k_3 = 4$.

So $\begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

3 mutually perpendicular eigenvectors are $\underset{(\lambda = -1)}{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}}$, $\underbrace{\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix}}_{\lambda = 2}$



p 105 Ex 3D Q 1 (a)(d)
2 (a) (b) (c)

(d) Diagonalisation

A DIAGONAL matrix has the form $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$. It is very easy to interpret geometrically.

Example 1 (cont)

We saw above that the matrix $M = \frac{1}{10}\begin{pmatrix} 11 & 3 \\ 3 & 19 \end{pmatrix}$ represents a stretch by s.f 2 in the direction of $\underline{b} = \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$.

In other words, if the axes were in the direction of $\underline{a}$ and $\underline{b}$, the matrix for the transformation would be $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, a diagonal matrix whose entries on the leading diagonal are the eigenvalues of M.

We could perform the transformation M in 3 stages

1. Rotate the plane so that $\underline{a} \to \underline{i}$ and $\underline{b} \to \underline{j}$, using a matrix P^{-1}.
2. Stretch in the y-direction using Λ.
3. Rotate back so that $\underline{i} \to \underline{a}$ and $\underline{j} \to \underline{b}$ using the matrix P.

What is the matrix P? Its first column is $\underline{a}$ and its second column is $\underline{b}$.

So the columns of P are the eigenvectors of M.

ie, $$P = \begin{pmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix}$$

We can generalise this, and extend it to 3D:

If M is a 3×3 matrix, then provided M has 3 eigenvectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ which are not all in the same plane (they need not be mutually perpendicular) then we can perform M in stages :–

1. Take $\underline{a}, \underline{b}, \underline{c}$ to $\underline{i}, \underline{j}, \underline{k}$ using P^{-1}
2. Stretch the axes using the eigenvalues of M (if an eigenvalue is −ve, the axis is also reflected), by using a diagonal matrix Λ
3. Take $\underline{i}, \underline{j}, \underline{k}$ back to $\underline{a}, \underline{b}, \underline{c}$ using P where P's columns are the eigenvectors of M in the order corresponding to the eigenvalues in Λ

So $$\boxed{M = P \Lambda P^{-1}}$$

or $P^{-1}MPP^{-1} = P^{-1}P \Lambda P^{-1} = \Lambda P^{-1}$

$P^{-1}MP = \Lambda P^{-1}P$

$$\boxed{P^{-1}MP = \Lambda}$$

(we say that 'P diagonalises M')

Not all matrices are diagonalisable.

SYMMETRIC matrices have the following property:–
We can always find a set of 3 mutually perpendicular unit eigenvectors $\underline{a}, \underline{b}, \underline{c}$, so can be diagonalised using a matrix P which is ORTHOGONAL. In this case P^{-1} is the same as P^T, so

$$\boxed{P^T M P = \Lambda}$$

Example 2 (cont)

For the matrix $M = \frac{1}{3}\begin{pmatrix} 5 & -2 & 2 \\ -2 & 2 & 4 \\ 2 & 4 & 2 \end{pmatrix}$ find a matrix P and a matrix Λ such that $P^T M P = \Lambda$.

In order for P to be orthogonal, we make the mutually perpendicular vectors found above into unit vectors and use them as the columns of P.

So $\Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1/3 & 2/\sqrt{5} & -2/\sqrt{45} \\ 2/3 & 0 & 5/\sqrt{45} \\ -2/3 & 1/\sqrt{5} & 4/\sqrt{45} \end{pmatrix}$

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