

Standard Integrals and Methods

We need to know all the C4 methods and when to use them.

We know some additional standard derivatives :-

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$+\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\operatorname{coth} x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \operatorname{coth} x$

For the integrals of $\tanh x$ etc, just remember the method :-

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$

$$\text{let } t = \cosh x \Rightarrow \frac{dt}{dx} = \sinh x \Rightarrow dx = \frac{dt}{\sinh x}$$

$$\begin{aligned} I &= \int \frac{1}{t} \, dt \\ &= \ln|t| \\ &= \underline{\underline{\ln|\cosh x| + C}} \end{aligned}$$

$$\int \operatorname{coth} x \, dx = \ln|\sinh x| + C \quad (\text{same method})$$

$$\int \operatorname{sech} x \, dx = \int \frac{2}{\cosh x} \, dx$$

$$= \int \frac{2}{e^x + e^{-x}} \, dx$$

$$= \int \frac{2e^x}{e^{2x} + 1} \, dx$$

$$\text{let } t = e^x \Rightarrow \frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{e^x}$$

$$I = \int \frac{2}{t^2 + 1} \, dt$$

$$\text{let } t = \tan \theta \Rightarrow \frac{dt}{d\theta} = \sec^2 \theta \quad dt = \sec^2 \theta \, d\theta$$

$$I = \int \frac{2}{\cancel{\tan^2 \theta + 1}} \cancel{\sec^2 \theta} \, d\theta$$

$$= 2\theta$$

$$= 2 \arctan t$$

$$= \underline{\underline{2 \arctan(e^x) + C}}$$

For hyperbolic integrals, either use the corresponding trig method, or use the definitions in terms of e^x .

p58 Ex 4A Q 1 beg, 2 bcef

p61 Ex 4B Q 1acd, 2abc, 3, 4, 5bc, 6, 7, 8, 10, 11

Trig and Hyperbolic SubstitutionsExample 1

$$\int \frac{1}{(x^2+9)^{3/2}} dx$$

$$\text{let } x = 3 \sinh t \quad \Rightarrow \quad \frac{dx}{dt} = 3 \cosh t$$

$$\Rightarrow dx = 3 \cosh t dt$$

$$I = \int \frac{1}{(9 \sinh^2 t + 9)^{3/2}} 3 \cosh t dt$$

$$= \int \frac{1}{(9 \cosh^2 t)^{3/2}} 3 \cosh t dt$$

$$= \int \frac{1}{27 \cosh^3 t} 3 \cosh t dt$$

$$= \frac{1}{9} \int \operatorname{sech}^2 t dt$$

$$= \frac{1}{9} \tanh t + C$$

$$= \frac{1}{9} \tanh \left(\operatorname{arsinh} \frac{x}{3} \right) + C$$

There are various ways of tackling similar questions. Also in the formula booklet there are 4 standard integrals of this type.

Example 2

$$\int \frac{1}{\sqrt{x^2-25}} dx$$

$$= \operatorname{arcosh} \left(\frac{x}{5} \right) + C$$

Example 3

$$\int \frac{1}{25 + 9x^2} dx$$

$$= \int \frac{1}{5^2 + (3x)^2} dx$$

which is $f(3x)$ where $f(x) = \int \frac{1}{5^2 + x^2} dx$.

$$= \frac{1}{3} \times \frac{1}{5} \arctan\left(\frac{3x}{5}\right) + C$$

$$= \underline{\underline{\frac{1}{15} \arctan\left(\frac{3x}{5}\right) + C}}$$

(or see p 64 Example 11)

Example 4

$$\int \frac{x+1}{\sqrt{x^2+1}} dx$$

Split into: -

$$= \int \frac{x}{\sqrt{x^2+1}} dx + \int \frac{1}{\sqrt{x^2+1}} dx$$

(A) (B)

For (A) let $t = x^2 + 1 \Rightarrow \frac{dt}{dx} = 2x$

$$dx = \frac{dt}{2x}$$

$$A = \int \frac{\cancel{x}}{\sqrt{t}} \frac{dt}{\cancel{2x}}$$

$$= \frac{1}{2} \int t^{-1/2} dt$$

$$= t^{1/2}$$

$$= \sqrt{x^2+1} + C$$

www.youtube.com/megalecture
www.megalecture.com

$$B = \text{arsinh } x$$

$$\text{So } I = \underline{\underline{\sqrt{x^2+1} + \text{arsinh } x + C}}$$

Ex 4c @ 1, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16

Example 5

$$\int \frac{1}{\sqrt{x^2 - 6x + 13}} dx$$

Complete the square:

$$= \int \frac{1}{\sqrt{(x-3)^2 + 4}} dx$$

$$= \underline{\underline{\text{arsinh} \left(\frac{x-3}{2} \right) + C}}$$

Ex 4D labceg, 2b, 4, 9, 10

Reduction Method of Integration

Example 1 If $I_n = \int_0^{\pi/2} \sin^n \theta d\theta$, show that

$$I_n = \frac{n-1}{n} I_{n-2}$$

Hence find $\int_0^{\pi/2} \sin^9 \theta d\theta$

$$I_n = \int_0^{\pi/2} \sin^n \theta \sin \theta \cos \theta \, d\theta$$

let $u = \sin^{n-1} \theta$ and $\frac{dv}{d\theta} = \sin \theta$

$$\frac{du}{d\theta} = (n-1) \sin^{n-2} \theta \cos \theta \quad v = -\cos \theta$$

$$I_n = \left[-\sin^{n-1} \theta \cos \theta \right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} \theta \cos^2 \theta \, d\theta$$

$$I_n = 0 + (n-1) \int_0^{\pi/2} \sin^{n-2} \theta (1 - \sin^2 \theta) \, d\theta$$

$$I_n = (n-1) \left[\int_0^{\pi/2} \sin^{n-2} \theta \, d\theta - \int_0^{\pi/2} \sin^n \theta \, d\theta \right]$$

$$I_n = (n-1) [I_{n-2} - I_n]$$

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

$$n I_n = (n-1) I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2} \quad \underline{\underline{QED.}}$$

$$\text{So } I_9 = \frac{8}{9} I_7$$

$$= \frac{8}{9} \frac{6}{7} I_5$$

$$= \frac{8}{9} \frac{6}{7} \frac{4}{5} \frac{2}{3} I_1$$

$$I_1 = \int_0^{\pi/2} \sin \theta \, d\theta = \left[-\cos \theta \right]_0^{\pi/2} = 0 - -1 = \underline{\underline{1}}$$

$$\underline{\underline{I_9 = \frac{128}{315}}}$$

Example 2

If $I_n = \int_0^1 x^n \sqrt{1-x^3} dx$, show

that $(2n+5)I_n = (2n-4)I_{n-3}$.

Hence find $\int_0^1 x^8 \sqrt{1-x^3} dx$

$$I_n = \int_0^1 x^{n-2} x^2 \sqrt{1-x^3} dx$$

let $u = x^{n-2}$ $\frac{dv}{dx} = x^2 \sqrt{1-x^3} dx$.

$$\frac{du}{dx} = (n-2)x^{n-3}$$

let $t = 1-x^3 \Rightarrow \frac{dt}{dx} = -3x^2$

$$\begin{aligned} v &= \int \cancel{x^2} \sqrt{t} \frac{dt}{-3\cancel{x^2}} \\ &= -\frac{1}{3} \times \frac{2}{3} t^{3/2} \\ &= -\frac{2}{9} (1-x^3)^{3/2} \end{aligned}$$

$$I_n = \left[-\frac{2}{9} x^{n-2} (1-x^3)^{3/2} \right]_0^1 + \frac{2}{9} (n-2) \int_0^1 x^{n-3} (1-x^3)^{3/2} dx$$

$$I_n = 0 + \frac{2}{9} (n-2) \int_0^1 x^{n-3} (1-x^3) \sqrt{1-x^3} dx$$

$$= \frac{2}{9} (n-2) \int_0^1 (x^{n-3} - x^n) \sqrt{1-x^3} dx$$

$$I_n = \frac{2}{9} (n-2) [I_{n-3} - I_n]$$

$$9I_n = (2n-4)I_{n-3} - (2n-4)I_n$$

$$(2n+5)I_n = (2n-4)I_{n-3}$$

QED

$$\begin{aligned} I_8 &= \frac{12}{21} I_5 \\ &= \frac{12}{21} \frac{6}{15} I_2 \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 x^2 \sqrt{1-x^3} dx \\ &= \left[-\frac{2}{9} (1-x^3)^{3/2} \right]_0^1 \\ &= 0 - \left(-\frac{2}{9} \right) = \frac{2}{9} \end{aligned}$$

$$I_8 = \frac{12}{21} \frac{6}{15} \frac{2}{9} = \underline{\underline{\frac{16}{315}}}$$

We don't always use parts

Example 3 If $I_n = \int_0^{\infty} \tan^n x dx$, show that

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$\begin{aligned} I_n &= \int_0^{\infty} \tan^2 x + \tan^{n-2} x dx \\ &= \int_0^{\infty} (\sec^2 x - 1) \tan^{n-2} x dx \\ &= \int_0^{\infty} \sec^2 x \tan^{n-2} x dx - I_{n-2} \end{aligned}$$

$$\begin{aligned} \text{let } t = \tan x &\Rightarrow \frac{dt}{dx} = \sec^2 x \\ &\Rightarrow dx = \frac{dt}{\sec^2 x} \end{aligned}$$

$$I_n = \int \frac{dx}{\cos^n x} - I_{n-2}$$
$$= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^x - I_{n-2}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

QED.

Ex 4F Q. 1, 2, 7, 10, 13, 15