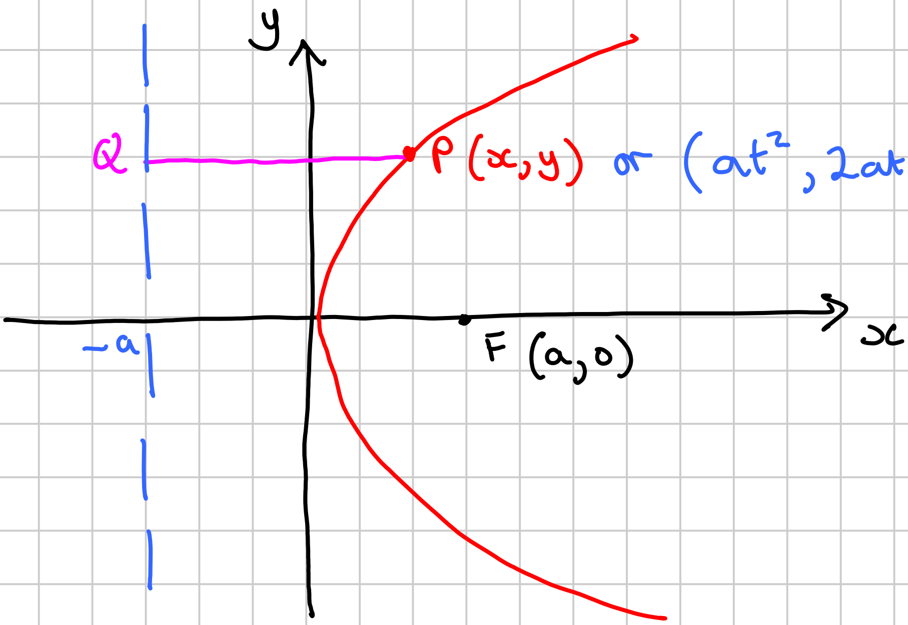
Coordinate Geometry – Conic Sections

If we slice a cone at various angles we can get
a circle
an ellipse
a parabola
a hyperbola
a pair of straight lines
These are the CONIC SECTIONS.

Parabola This can also be defined as the locus of points equidistant from a given point (the focus F) and a given line (the directrix)

The standard parabola has focus $(a, 0)$ and directrix $x = -a$



To find its equation, we want $PF = PQ$

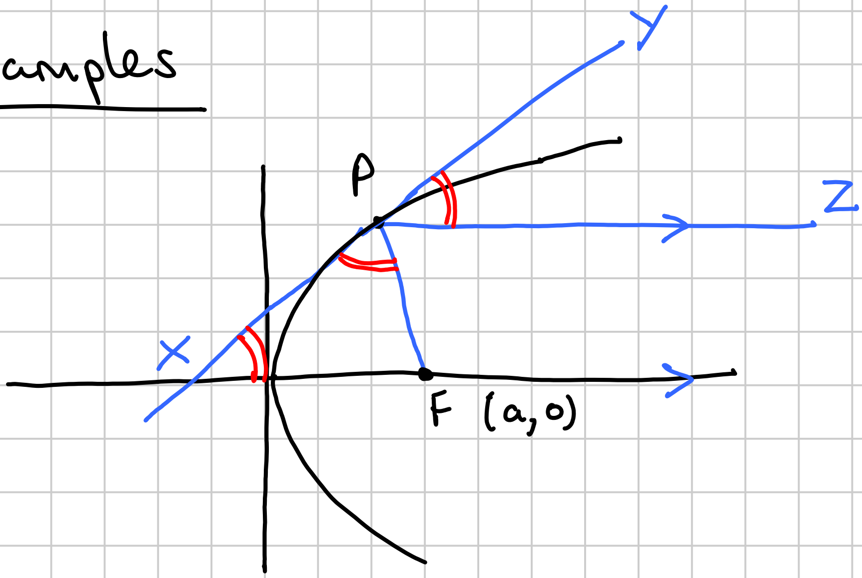
$$\sqrt{(x-a)^2 + y^2} = x + a$$
$$(x-a)^2 + y^2 = (x+a)^2$$
$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$
$$\boxed{y^2 = 4ax}$$

or in parametric form $\boxed{\begin{aligned} x &= at^2 \\ y &= 2at \end{aligned}}$

p 83 Ex 4.1 1, 4, 5, 6, 7
p 91 4.3 2, 5, 9, 10

Examples

1



Prove that $ZPY = FPX$

Since $ZPY = PXF$, we can prove this by showing that $FP = FX$, so that PFX is an isosceles triangle

Let P be $(ap^2, 2ap)$

$$\left.\begin{aligned} x = ap^2 &\Rightarrow \frac{dx}{dp} = 2ap \\ y = 2ap &\Rightarrow \frac{dy}{dp} = 2a \end{aligned}\right\} \quad \frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx}$$

$$= 2a \times \frac{1}{2ap} = \underline{\underline{\frac{1}{p}}}$$

Equation of tangent is

$$y - 2ap = \frac{1}{p}(x - ap^2)$$

$$py - 2ap^2 = x - ap^2$$

$$py = x + ap^2$$

At X, $y = 0 \Rightarrow x = -ap^2$

$$FX = a + ap^2$$

$$FP = \sqrt{(ap^2 - a)^2 + (2ap - 0)^2}$$

$$= \sqrt{a^2p^4 - 2a^2p^2 + a^2 + 4a^2p^2}$$

$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$

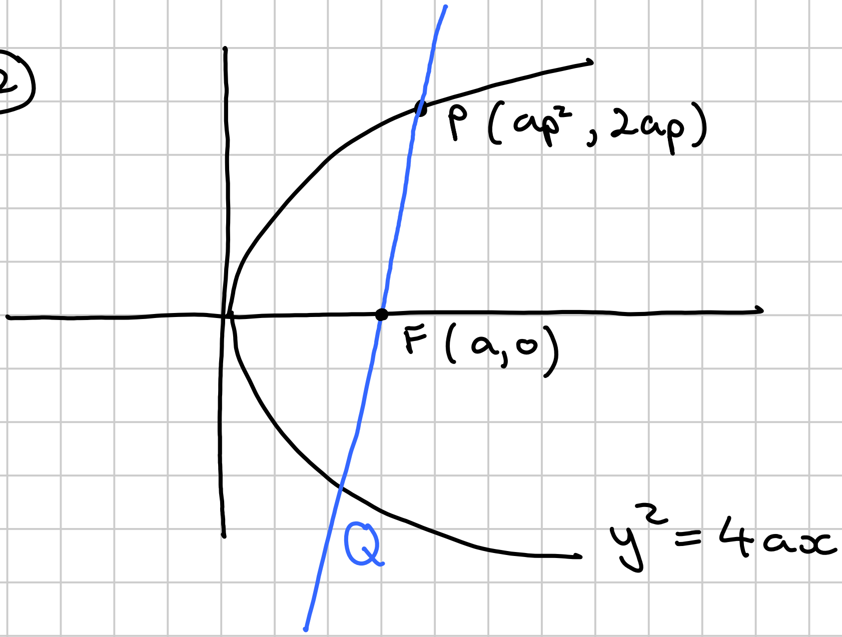
$$= \sqrt{(ap^2 + a)^2}$$

$$= ap^2 + a$$

$$= FX$$

QED

2



The line PF meets the parabola again at Q $(aq^2, 2aq)$. Find q in terms of p.

$$\text{Gradient PF} = \frac{2ap}{ap^2 - a} = \frac{2p}{p^2 - 1}$$

Equation of PF $\quad y - 0 = \frac{2p}{p^2-1}(x - a)$

$$(p^2 - 1)y = 2px - 2ap$$

To find where this meets the curve, substitute in the parametric equations :-

$$(p^2 - 1)\cancel{2}\cancel{a}q = \cancel{2}p\cancel{a}q^2 - \cancel{2}\cancel{a}p$$

$$p^2q - q = pq^2 - p$$

$$p^2q - pq^2 = q - p$$

$$pq(p-q) = q - p$$

$$= -(p-q)$$

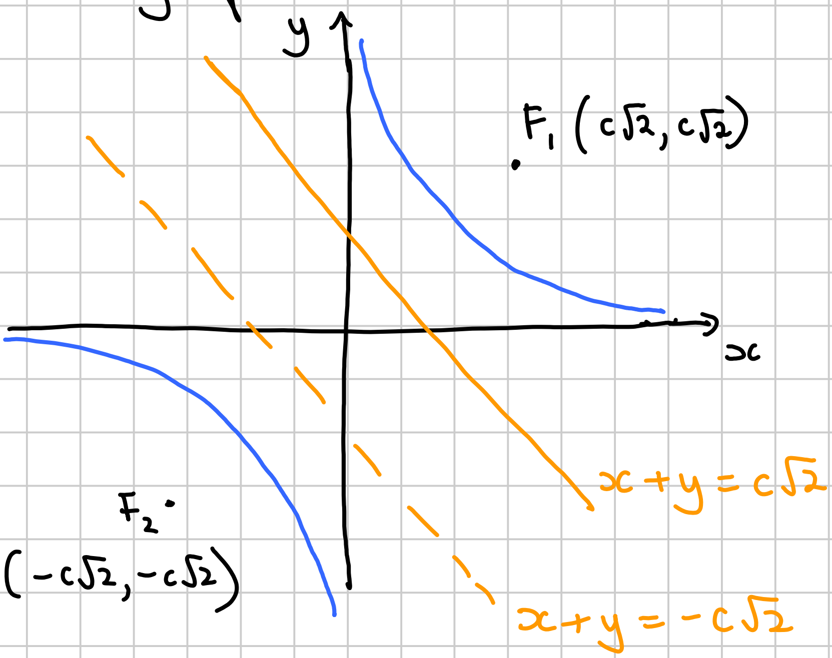
Divide by $(p-q)$ [since $p - q \neq 0$ since Q is a different point to P]

$$pq = -1$$

$$\underline{\underline{q = \frac{-1}{p}}}$$

Rectangular Hyperbola

A hyperbola has two asymptotes. If it is 'rectangular', the asymptotes are at right angles. So it is convenient to make the axes the asymptotes.

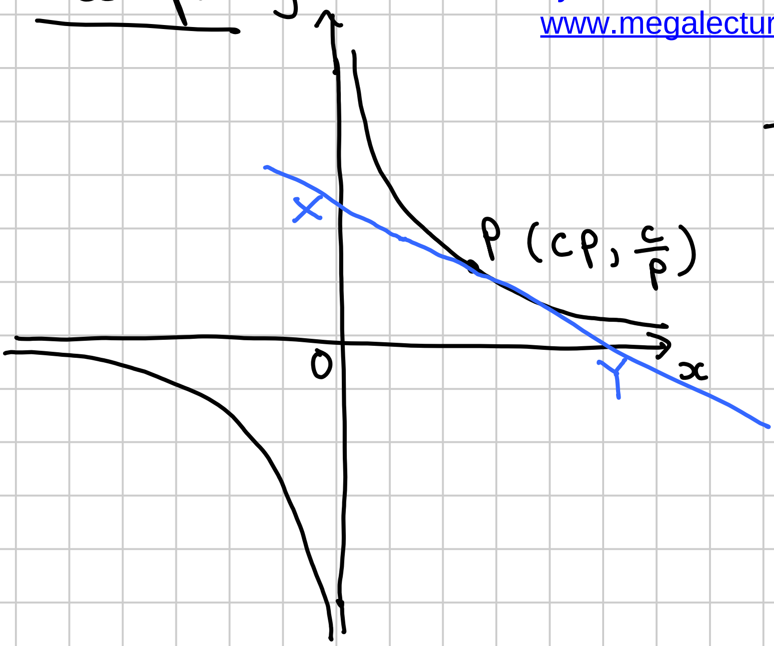


} directrices

$$\boxed{y = \frac{c^2}{x} \quad \text{or} \quad xy = c^2}$$

$$\boxed{\begin{aligned} x &= ct \\ y &= \frac{c}{t} \end{aligned}}$$

Example



P is a general point on the curve $xy = c^2$.

The tangent at P meets the asymptotes at X and Y. Prove that the area of triangle XOY is constant whatever the position of P.

$$\left.\begin{aligned} x = ct &\Rightarrow \frac{dx}{dt} = c \\ y = \frac{c}{t} &\Rightarrow \frac{dy}{dt} = -\frac{c}{t^2} \end{aligned}\right\} \quad \frac{dy}{dx} = -\frac{1}{t^2}$$

At P with parameter p, gradient of tangent is $\frac{-1}{p^2}$

Eqn of tangent $\quad y - \frac{c}{p} = \frac{-1}{p^2}(x - cp)$

$$p^2 y - cp = -x + cp$$

$$p^2 y + x = 2cp$$

At X, $x = 0$, so $y = \frac{2c}{p}$

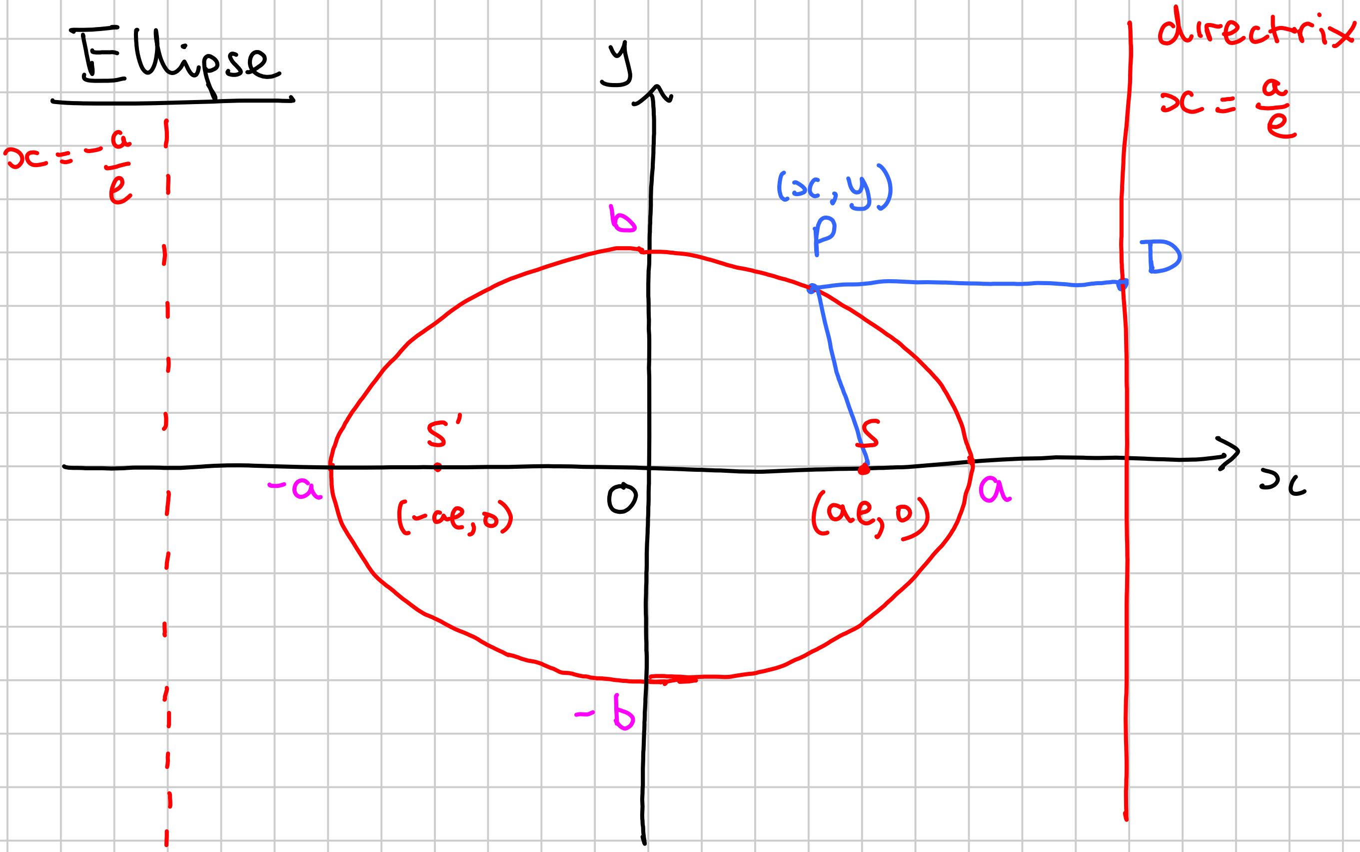
At Y, $y = 0$, so $x = 2cp$

So area of $\Delta XOY = \frac{1}{2} \times 2cp \times \frac{2c}{p}$

$= 2c^2$ which is independent of p

p 87 Ex 4.2 Q 3, 4 ac, 5, 6, 8
p 91 Ex 4.3 Q 4, 6, 8
p 96 Ex 4.4 Q 3, 9

Ellipse



An ellipse is a locus of points P such that PS = e PD where e is the eccentricity and $0 < e < 1$

$$PS = e\,PD$$

$$\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right)$$

$$x^2 - 2aex + a^2e^2 + y^2 = e^2\left(\frac{a^2}{e^2} - 2\frac{a}{e}x + x^2\right)$$

$$= a^2 - 2aex + e^2x^2$$

$$(1 - e^2)x^2 + y^2 = (1 - e^2)a^2$$

(Divide both sides by $a^2(1-e^2)$)

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Now let $b^2 = a^2(1-e^2)$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

STANDARD CARTESIAN EQUATION.

If $y = 0$, $x = \pm a$
If $x = 0$, $y = \pm b$

If $e = 0$, $b = a$ and the equation becomes $x^2 + y^2 = a^2$ which is a circle.

The standard parametric equations for an ellipse are

$$\boxed{\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned}}$$

Examples

① Find the equation of the tangent at $P(a\cos\theta, b\sin\theta)$.
This tangent meets the x-axis at X and the y-axis at Y.
Find the equation of the locus of M, the midpoint of XY.

$x = a\cos\theta \Rightarrow \frac{dx}{d\theta} = -a\sin\theta$

$y = b\sin\theta \Rightarrow \frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{b\cos\theta}{-a\sin\theta}$$

Equ of tangent $y - b\sin\theta = \frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$ay\sin\theta + bx\cos\theta = ab$

At X, $y = 0 \Rightarrow x = \frac{a}{\cos\theta}$ $\quad X\left(\frac{a}{\cos\theta}, 0\right)$

At Y $x = 0 \Rightarrow y = \frac{b}{\sin\theta}$ $\quad Y\left(0, \frac{b}{\sin\theta}\right)$

So M has coordinates $\left(\frac{a}{2\cos\theta}, \frac{b}{2\sin\theta}\right)$

We need to eliminate θ from $x = \frac{a}{2\cos\theta}$

$y = \frac{b}{2\sin\theta}$

$$\Rightarrow \left.\begin{aligned}\cos\theta &= \frac{a}{2x}\\ \sin\theta &= \frac{b}{2y}\end{aligned}\right\} \Rightarrow \left(\frac{a}{2x}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$$

p 24 Ex 2A Q 1 a (i)(ii), b(i)(ii)

p 27 Ex 2B Q 1a, 2, 4, 5, 9

② The line $x + y = k$ $(k > 0)$ is a tangent to the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

Find the value of k and the coordinates of the point P where the line touches the ellipse.

To find where the line meets the ellipse:

$$y = k - x$$

Substitute:

$$\frac{x^2}{16} + \frac{(k-x)^2}{9} = 1$$

$$9x^2 + 16\left(k^2 - 2kx + x^2\right) = 144$$

$$25x^2 - 32kx + 16k^2 - 144 = 0$$

If the line is a tangent, this has only one solution so $b^2 - 4ac = 0$

$$1024k^2 - 100\left(16k^2 - 144\right) = 0$$

$$576k^2 = 14400$$

$$k^2 = 25$$

$$k = \pm 5$$

but $k > 0$ so $k = 5$

So $25x^2 - 160x + 256 = 0$

$$(5x - 16)^2 = 0$$

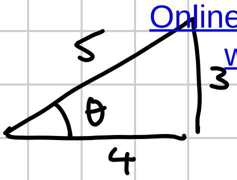
$$x = \frac{16}{5} \qquad y = \frac{9}{5} \qquad \underline{\underline{P\left(\frac{16}{5}, \frac{9}{5}\right)}}$$

Alternatively :– P has coords $(4\cos\theta, 3\sin\theta)$ for some value of θ.

$$\text{Gradient at P} = \frac{3\cos\theta}{-4\sin\theta}$$

Gradient of $x + y = k$ is -1, so $\dfrac{3\cos\theta}{-4\sin\theta} = -1$

$$\tan\theta = \frac{3}{4}$$



$\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$

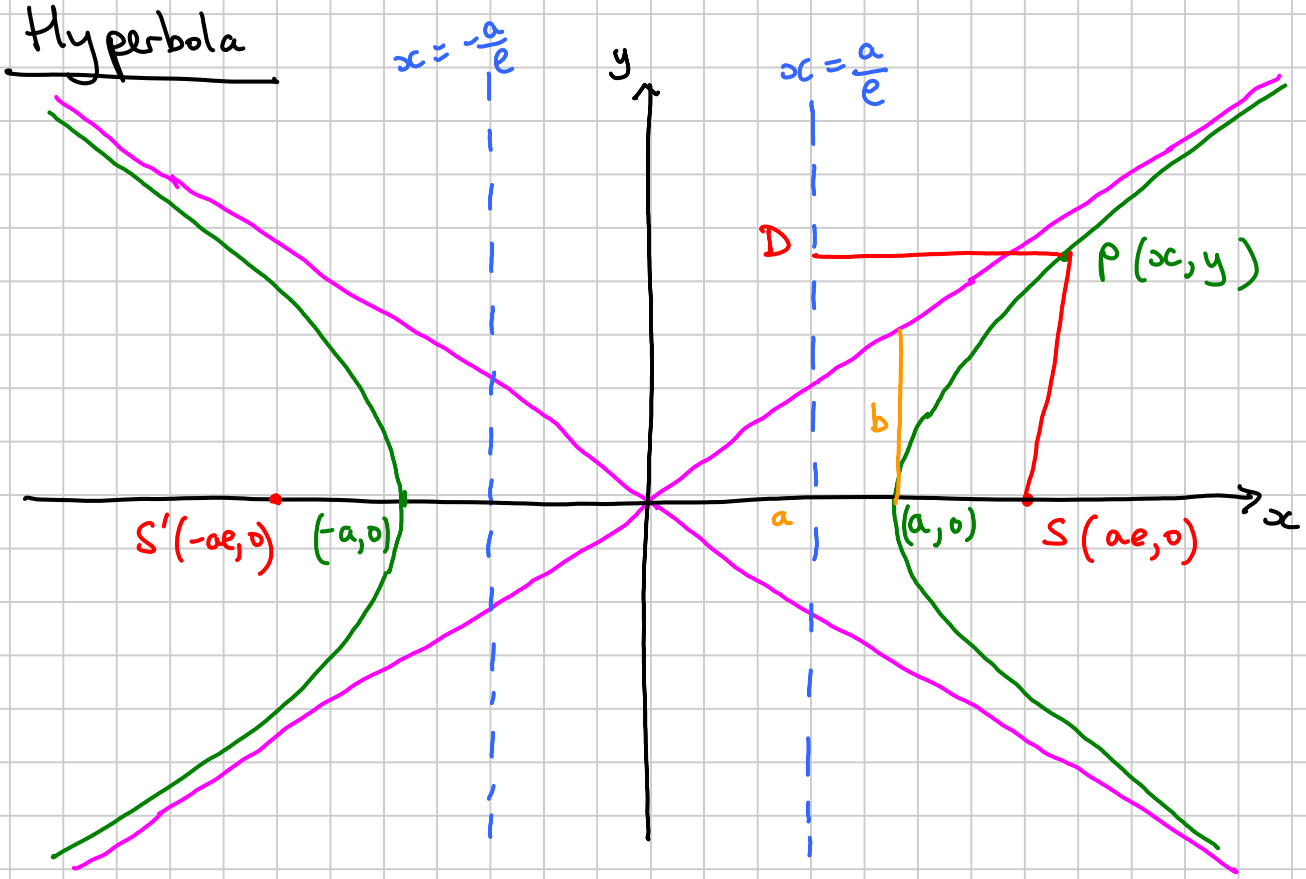
$P\left(\frac{16}{5}, \frac{9}{5}\right)$

If P lies on $x + y = k$, $\quad k = \frac{16}{5} + \frac{9}{5} = 5$

p 27 Ex 2B Q 6, 8

p 39 Ex 2E Q 1 ac, 2, 3, 4, 7, 8

Hyperbola



Same definition as for an ellipse, except that $e > 1$

$$PS = e\,PD$$

(same working as for ellipse $\Longrightarrow$)

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

Since $1-e^2$ is $-$ve, write this as

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$$

Let $\boxed{b^2 = a^2(e^2-1)}$

$$\Rightarrow \boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$$

Write this as $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$

Now as $x \to \infty$, the "-1" becomes negligible,

so
$$y^2 \approx \frac{b^2}{a^2}x^2$$
$$y \approx \pm\frac{b}{a}x$$

So the asymptotes are $\boxed{y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x}$

If $b = a$, the asymptotes are $y = \pm x$ which are at right angles, so we have a rectangular hyperbola.

In this case
$$a^2(e^2-1) = a^2$$
$$e^2 - 1 = 1$$
$$e = \sqrt{2}$$

There are two common parametric equations:

$\left.\begin{aligned} x &= a\cosh\theta \\ y &= b\sinh\theta \end{aligned}\right\}$ but this only gives the +ve branch of the hyperbola $(-\infty < \theta < \infty)$

$\left.\begin{aligned} x &= a\sec\theta \\ y &= b\tan\theta \end{aligned}\right\}$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ gives the +ve branch
$\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ gives the $-$ve branch

Example The tangent at P $(a\sec\theta, b\tan\theta)$ meets the asymptotes at Q and R. Prove that the area of triangle OQR is constant.

$$x = a\sec\theta \Rightarrow \frac{dx}{d\theta} = a\sec\theta\tan\theta$$

$$y = b\tan\theta \Rightarrow \frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\frac{1}{\cos\theta}\frac{\cos\theta}{\sin\theta} = \frac{b}{a\sin\theta}$$

Eqn of tangent

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

$$ay\sin\theta - ab\frac{\sin^2\theta}{\cos\theta} = bx - ab\sec\theta$$

$$ay\sin\theta\cos\theta - ab\sin^2\theta = bx\cos\theta - ab$$

$$ay\sin\theta\cos\theta - bx\cos\theta + ab\cos^2\theta = 0$$

$$ay\sin\theta - bx + ab\cos\theta = 0$$

Meets $y = \frac{b}{a}x$ when

$$\cancel{b}x\sin\theta - \cancel{b}x + a\cancel{b}\cos\theta = 0$$

$$a\cos\theta = x(1-\sin\theta)$$

$$\left.\begin{aligned} x &= \frac{a\cos\theta}{1-\sin\theta} \\ y &= \frac{b\cos\theta}{1-\sin\theta} \end{aligned}\right\} Q$$

Meets $y = -\frac{b}{a}x$ when

$$-bx\sin\theta - bx + ab\cos\theta = 0$$

$$\left.\begin{aligned} x &= \frac{a\cos\theta}{1+\sin\theta} \\ y &= \frac{-b\cos\theta}{1+\sin\theta} \end{aligned}\right\} R$$

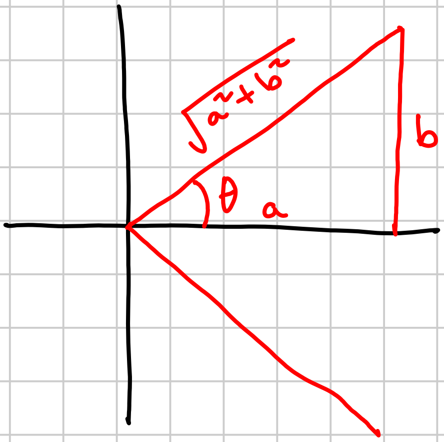
Distance

$$OQ = \sqrt{\frac{a^2\cos^2\theta}{(1-\sin\theta)^2} + \frac{b^2\cos^2\theta}{(1-\sin\theta)^2}}$$

$$= \sqrt{\frac{\cos^2\theta}{(1-\sin\theta)^2}(a^2+b^2)}$$

$$= \left(\frac{\cos\theta}{1-\sin\theta}\right)\sqrt{a^2+b^2}$$

$$OR = \left(\frac{\cos\theta}{1+\sin\theta}\right)\sqrt{a^2+b^2}$$



$$\tan\theta = \frac{b}{a}.$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}} \qquad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin 2\theta = \frac{2ab}{a^2+b^2}$$

Area OQR

$$= \frac{1}{\cancel{2}}\left(\frac{\cos\theta}{1-\sin\theta}\cancel{\sqrt{a^2+b^2}}\right)\left(\frac{\cos\theta}{1+\sin\theta}\cancel{\sqrt{a^2+b^2}}\right)\left(\frac{\cancel{2}ab}{\cancel{a^2+b^2}}\right)$$

$$= \frac{ab\,\cancel{\cos^2\theta}}{\cancel{1-\sin^2\theta}}$$

$$= \underline{\underline{ab}}$$

P 33 Ex 2D Q 1a, 4, 5, 6

P 40 Ex 2E Q 5ac, 6a

P