

Chapter 3 - Complex Numbers

We know that a complex number z can be written as

$$z = r(\cos \theta + i \sin \theta) \quad (r = \text{modulus}, \theta = \text{argument})$$

We can now find a more compact way of writing this.

Consider $e^{i\theta}$. Using the power series for e^x , this is

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \\ &\quad + i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right] \\ &= \cos \theta + i \sin \theta \end{aligned}$$

So an alternative way of writing $z = r(\cos \theta + i \sin \theta)$
is $z = r e^{i\theta}$

This makes it obvious that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$:
if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

It also leads to de Moivre's theorem :-

$$\begin{aligned} (\cos \theta + i \sin \theta)^n &= (e^{i\theta})^n \\ &= e^{i(n\theta)} \\ &= \cos(n\theta) + i \sin(n\theta) \end{aligned}$$

It also gives us the links between trig and hyperbolic functions :-

$$e^{i\theta} = \cos \theta + i \sin \theta \quad (1)$$

$$\begin{aligned} e^{-i\theta} &= \cos(-\theta) + i \sin(-\theta) \\ &= \cos \theta - i \sin \theta \quad (2) \end{aligned}$$

$$(1) + (2) \div 2 \Rightarrow \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh(i\theta)$$

$$(1) - (2) \div 2 \Rightarrow i \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \sinh(i\theta)$$

Replacing θ by $i\theta$ in these we get
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$$\cos i\theta = \cosh(i^2\theta) = \cosh(-\theta) = \cosh \theta$$

$$\text{and } i\sin i\theta = \sinh(i^2\theta) = \sinh(-\theta) = -\sinh \theta$$

(multiply by i)
 So

$$-\sin i\theta = -i\sinh \theta$$

$$\cos(i\theta) = \cosh \theta$$

$$\sin(i\theta) = i\sinh \theta$$

This enables us to use trig identities to find hyperbolic identities, and explains Osborne's rule.

e.g ① From $\sin 2\theta = 2\sin\theta \cos\theta$

(let $\theta = ix$)

$$\sin(2ix) = 2\sin(ix)\cos(ix)$$

$$i\sinh 2x = 2i\sinh x \cosh x$$

($\div i$)

$$\sinh 2x = 2\sinh x \cosh x$$

② From

$$\cos^2 \theta + \sin^2 \theta = 1$$

(let $\theta = ix$)

$$\cos^2(ix) + \sin^2(ix) = 1$$

$$(\cosh ix)^2 + (i\sinh ix)^2 = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

We can have logs of negative numbers

e.g. to find $\ln(-2)$:

$$\arg(-2) = \pi, \text{ so write } -2 = 2e^{i\pi}$$

$$\ln(-2) = \ln(2e^{i\pi})$$

$$= \ln 2 + \ln(e^{i\pi})$$

$$= \ln 2 + i\pi$$

We can solve 'impossible' trig equations

e.g

$$\cos z = \frac{5}{3} \quad (> 1!)$$

$$\Rightarrow \cosh(iz) = \frac{5}{3}$$

$$iz = \operatorname{arccosh} \frac{5}{3}$$

$$= \pm \ln \left(\frac{5}{3} + \sqrt{2\frac{5}{3} - 1} \right)$$

$$= \pm \ln 3$$

$$z = \pm i \ln 3$$

Find i^i : $|i|=1$, $\arg(i) = \frac{\pi}{2}$ so $i = e^{i\pi/2}$. So $i^i = (e^{i\pi/2})^i = e^{-\pi/2} = 0.2078\dots$

We can also see Euler's formula result linking the 5 most fundamental numbers:

$$e^{i\pi} + 1 = \cos\pi + i\sin\pi + 1$$

$$= -1 + 0 + 1$$

$$e^{i\pi} + 1 = 0$$

Using real and imaginary parts

$$\operatorname{Re}(x + iy) = x, \quad \operatorname{Im}(x + iy) = y$$

Take care:

$$\operatorname{Re}(z_1 \pm z_2) = \operatorname{Re}(z_1) \pm \operatorname{Re}(z_2)$$

$$\text{BUT } \operatorname{Re}(z_1 z_2) \neq \operatorname{Re}(z_1) \operatorname{Re}(z_2)$$

Trig Identities

We can use de Moivre's theorem to find various trig identities

① Deriving trig identities for sin, cos, tan of multiple angles.

e.g. Express $\cos 6\theta$ in terms of powers of $\cos \theta$
Hence solve the equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$

$$\cos 6\theta = \operatorname{Re}(\cos 6\theta + i\sin 6\theta)$$

$$\Rightarrow \cos 6\theta = \operatorname{Re}(\cos \theta + i\sin \theta)^6$$

(write $\cos \theta = c$ and $\sin \theta = s$)

$$= \operatorname{Re}(c^6 + 6c^5is + 15c^4(is)^2 + 20c^3(is)^3 + 15c^2(is)^4 + 6c(is)^5 + (is)^6)$$

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$$= c^6 - 15c^4 + 15c^2 - 1$$

$$= c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$$

$$= c^6 - 15c^4 + 15c^6 + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$$

$$= 32c^6 - 48c^4 + 18c^2 - 1$$

So $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$

To solve $32x^6 - 48x^4 + 18x^2 - 1 = 0$

Let $x = \cos \theta \Rightarrow \cos 6\theta = 0$

$$6\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \frac{11\pi}{2}$$

$$\theta = \frac{\pi}{12} \text{ or } \frac{\pi}{4} \text{ or } \frac{5\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{3\pi}{4} \text{ or } \frac{11\pi}{12}$$

$$x = \left(\cos \frac{\pi}{12} \text{ or } \cos \frac{11\pi}{12} \right) \text{ or } \left(\frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}} \right) \text{ or } \left(\cos \frac{5\pi}{12} \text{ or } \cos \frac{7\pi}{12} \right)$$

$$= \pm \cos \frac{\pi}{12} \text{ or } \pm \frac{1}{\sqrt{2}} \text{ or } \pm \cos \frac{5\pi}{12}$$

② Deriving trig identities for powers of sin and cos.

We showed that

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

In the same way we can show that

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2} \quad \text{and} \quad \sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

Example : Express $\sin^4 \theta$ in terms of multiple angles and hence find $\int \sin^4 \theta \, d\theta$

$$\sin^4 \theta = \left[\frac{e^{i\theta} - e^{-i\theta}}{2i} \right]^4$$

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$$\begin{aligned}
 &= \frac{1}{16} \left[e^{i4\theta} + 4e^{i2\theta}(-e^{-i\theta})^2 + 6e^{i2\theta}(-e^{-i\theta})^3 + (-e^{-i\theta})^4 \right] \\
 &= \frac{1}{16} \left[e^{i4\theta} - 4e^{i2\theta} + 6 - 4e^{-i2\theta} + e^{-i4\theta} \right] \\
 &= \frac{1}{16} \left[(e^{i4\theta} + e^{-i4\theta}) - 4(e^{i2\theta} + e^{-i2\theta}) + 6 \right] \\
 &= \frac{1}{16} \left[2\cos 4\theta - 4(2\cos 2\theta) + 6 \right] \\
 &= \frac{1}{8} \left[\cos 4\theta - 4\cos 2\theta + 3 \right]
 \end{aligned}$$

Here $\int \sin^4 \theta \, d\theta = \frac{1}{8} \int \cos 4\theta - 4\cos 2\theta + 3 \, d\theta$

$$= \frac{1}{8} \left[\frac{1}{4} \sin 4\theta - 2\sin 2\theta + 3\theta \right] + c$$

Ex 3.4 Q 2, 5, 6, 7

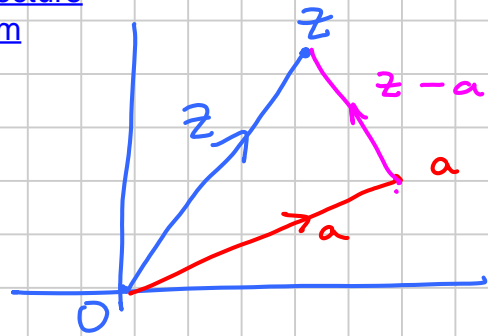
Loci in the Argand Diagram

An equation involving z is satisfied by a set of points (a locus).

We can sometimes see this geometrically, but if we can't we can fall back to writing $z = x + iy$ and finding a Cartesian equation for the locus.

Examples

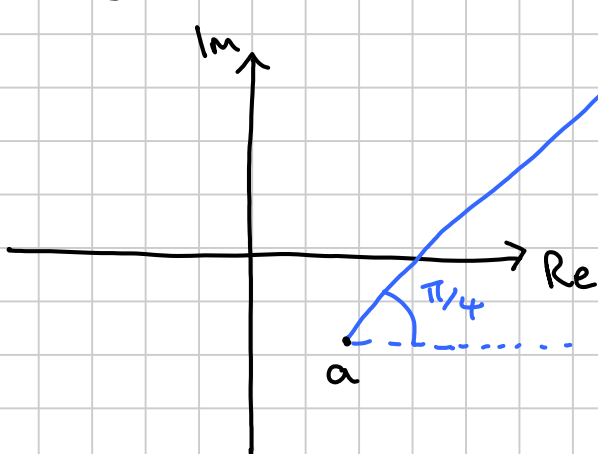
① $z - a$ can be represented by a vector from the point a to the point z .



So $|z - a|$ is the distance from a to z .

So the equation $|z - a| = 3$ is a circle, centre a and radius 3.

② $\arg(z - a) = \frac{\pi}{4}$

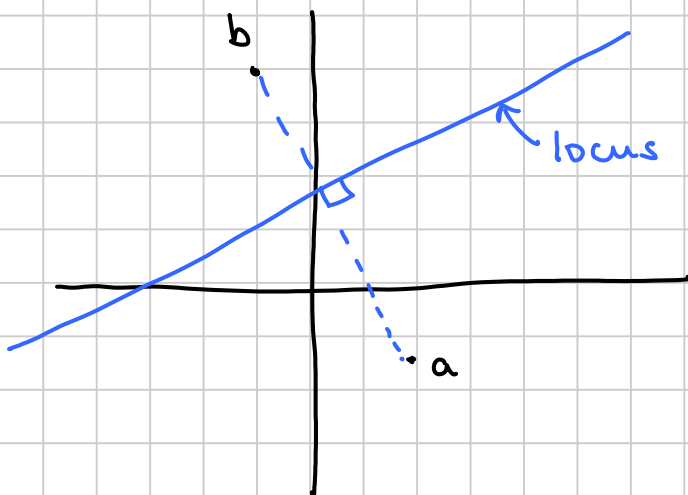


The locus is a 'half-line' or 'ray'.

③ $\left| \frac{z - a}{z - b} \right| = 1$

This can be written as $\frac{|z - a|}{|z - b|} = 1$

or $|z - a| = |z - b|$

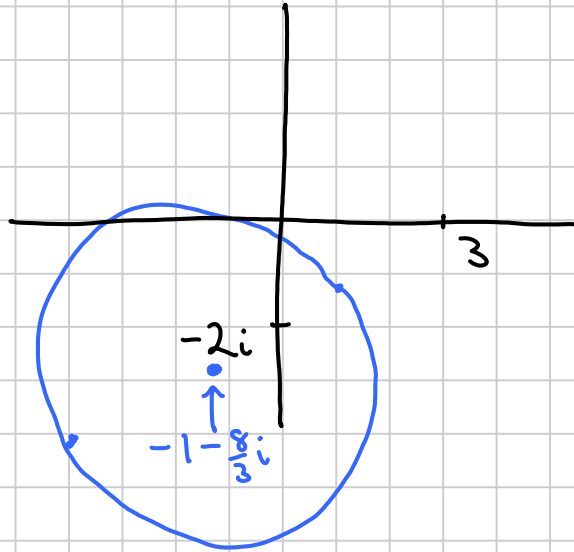


locus is the perpendicular bisector of the line joining a and b

(4)

$$\left| \frac{z-3}{z+2i} \right| = 2 \quad \left| \frac{z-3}{z+2i} \right| = 2 \quad \left| z-3 \right| = 2 \left| z+2i \right|$$

$$= 2 \left| z - (-2i) \right|$$



Write this as

$$\left| x+iy-3 \right| = 2 \left| x+iy+2i \right|$$

$$\Rightarrow \left| (x-3)+iy \right| = 2 \left| x+i(y+2) \right|$$

$$\Rightarrow \sqrt{(x-3)^2 + y^2} = 2 \sqrt{x^2 + (y+2)^2}$$

$$\Rightarrow (x-3)^2 + y^2 = 4(x^2 + (y+2)^2)$$

$$\Rightarrow x^2 - 6x + 9 + y^2 = 4(x^2 + y^2 + 4y + 4)$$

$$0 = 3x^2 + 3y^2 + 6x + 16y + 7$$

$$0 = x^2 + y^2 + 2x + \frac{16}{3}y + \frac{7}{3}$$

$$0 = (x+1)^2 - 1 + \left(y + \frac{8}{3}\right)^2 - \frac{64}{9} + \frac{7}{3}$$

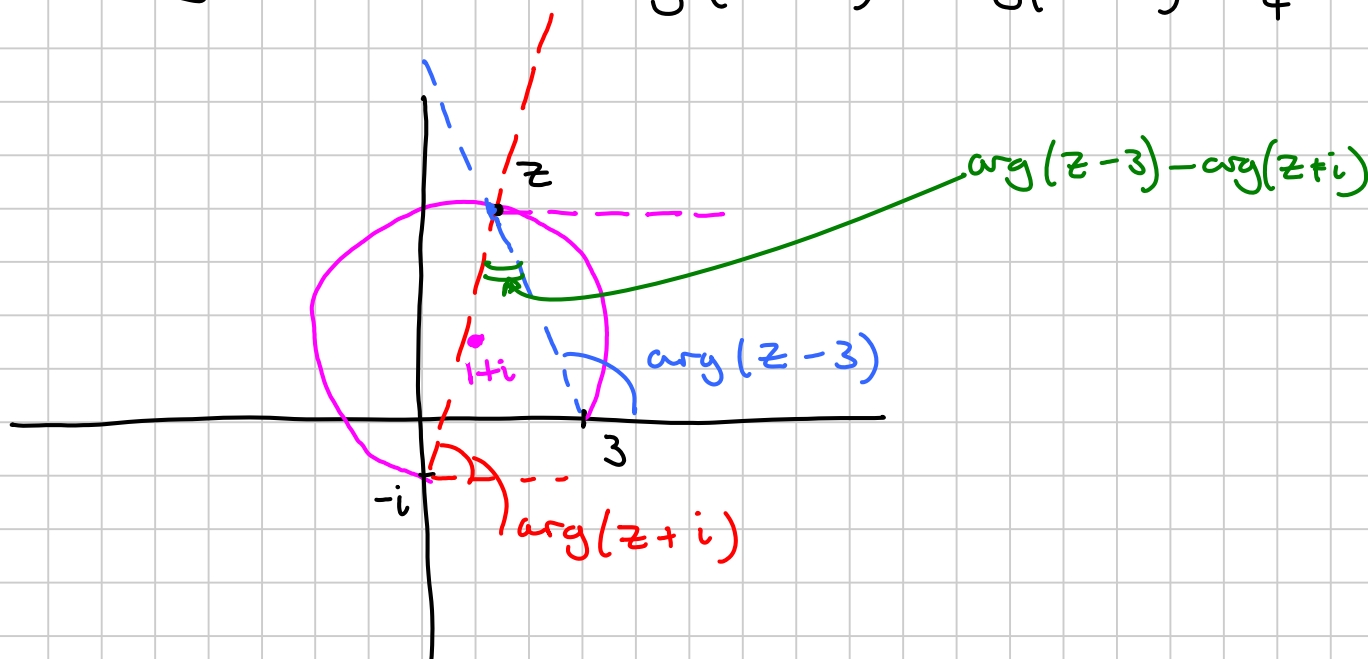
$$\frac{52}{9} = (x+1)^2 + \left(y + \frac{8}{3}\right)^2$$

which is a circle, centre $\left(-1, -\frac{8}{3}\right)$ radius $\frac{\sqrt{52}}{3}$.

5

$$\arg \left(\frac{z-3}{z+i} \right) = \frac{\pi}{4}$$

Geometrically : write as $\arg(z-3) - \arg(z+i) = \frac{\pi}{4}$



By the theorem about angles in the same segment, the locus is the major arc of a circle through 3 and $-i$.

Algebraically : let $z = x + yi$

$$\arg \left(\frac{(x-3) + yi}{x + (y+1)i} \right) = \frac{\pi}{4}$$

$$\arg \left(\frac{[(x-3) + yi][x - (y+1)i]}{[x + (y+1)i][x - (y+1)i]} \right) = \frac{\pi}{4}$$

$$\arg \left(\frac{x(x-3) + y(y+1) + i[x y - (x-3)(y+1)]}{x^2 + (y+1)^2} \right) = \frac{\pi}{4}$$

$$\arg \left(\frac{x^2 - 3x + y^2 + y}{x^2 + (y+1)^2} + i \frac{3y - x + 3}{x^2 + (y+1)^2} \right) = \frac{\pi}{4}$$

$$\frac{3y - x + 3}{x^2 - 3x + y^2 + y} = \tan \left(\frac{\pi}{4} \right) = 1$$

$$3y - x + 3 = x^2 - 2xc + y^2 + y$$

$$0 = x^2 - 2xc + y^2 - 2y - 3$$

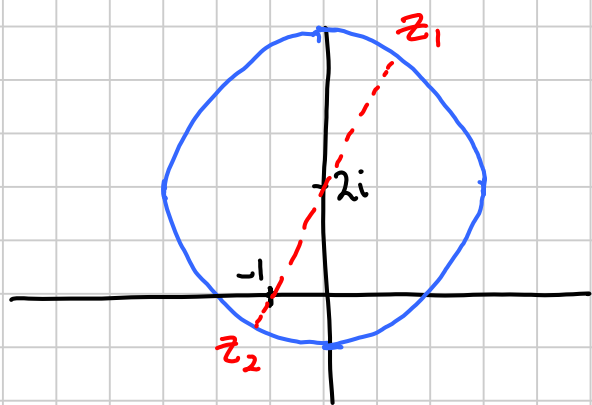
$$0 = (x-1)^2 - 1 + (y-1)^2 - 1 - 3$$

$$5 = (x-1)^2 + (y-1)^2$$

Which is a circle centre $(1, 1)$ or $1+i$ and radius $\sqrt{5}$.

(but from the geometric interpretation we only want the major arc of this circle)

(6) If $|z - 2i| = 3$ find the greatest and least values of $|z + 1|$



z must lie on the blue circle.

$|z + 1|$ is the distance from z to -1

$|z + 1|$ is a maximum when $z = z_1$ and is equal to

$$\sqrt{1^2 + 2^2} + 3 = \underline{\underline{3 + \sqrt{5}}}$$

$|z + 1|$ is a minimum when $z = z_2$ and is equal to

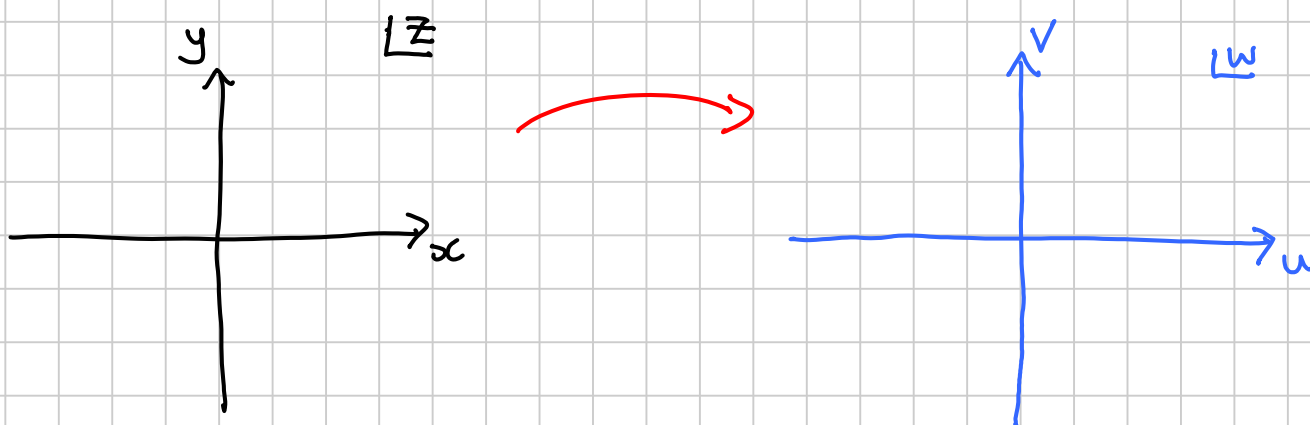
$$\underline{\underline{3 - \sqrt{5}}}$$

p 51 Ex 3.5 Q 1bc, 2bc, 3bcd, 4, 5abc, 6abde, 7, 11, 12, 13

p 56 Ex 3.6 Q 2ce, 3, 4, 6, 8

Transformations of the Complex Plane

If we have an equation $w = f(z)$ where z and w are both complex, we can visualize this as a transformation of the z -plane into the w -plane.



Some common transformations :

$$w = z + c$$

Translation $\begin{pmatrix} x \\ y \end{pmatrix}$ where $c = x + yi$

$$w = -z$$

Rotation π about 0

$$w = z^*$$

Reflection in the real axis

$$w = -z^*$$

Reflection in the imaginary axis

$$w = iz$$

Rotation $\frac{\pi}{2}$ (anticlockwise)

$$w = e^{i\theta} z$$

Rotation θ

$$w = kz$$

Enlargement s.f k , centre 0.

For most other transformations, it is hard to visualize the transformation as a whole. However, we can see or work out the image of a particular locus.

Principles

- Use geometric visualisation where possible
- Work with z and w where possible, but switch to x, y, u and v if necessary.

- For most equations it is useful to make z the subject first.

Examples

① Find the image of the arc $\arg\left(\frac{z}{z+i}\right) = \frac{1}{4}\pi$ under the transformation $w = \frac{z}{1-iz}$.

Change the subject

$$w(1-iz) = z$$

$$w - izw = z$$

$$w = z + izw$$

$$w = z(1+iw)$$

$$z = \frac{w}{1+iw}$$

Substitute into locus

$$\arg\left(\frac{\frac{w}{1+iw}}{\frac{w}{1+iw} + i}\right) = \frac{1}{4}\pi$$

Multiply top and bottom by $1+iw$

$$\arg\left(\frac{w}{w + i(1+iw)}\right) = \frac{1}{4}\pi$$

$$\arg\left(\frac{w}{w + i - w}\right) = \frac{1}{4}\pi$$

$$\arg(w) - \arg(i) = \frac{1}{4}\pi$$

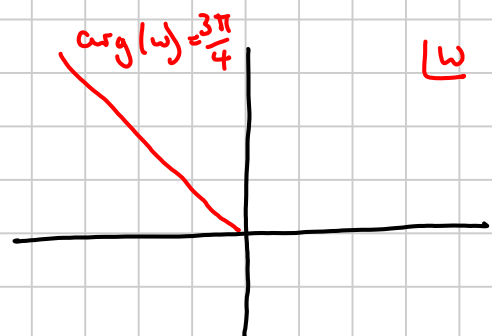
$$\arg(w) - \frac{\pi}{2} = \frac{1}{4}\pi$$

$$\arg(w) = \frac{3\pi}{4}$$

$$\arg\left(\frac{z}{z+i}\right) = \frac{\pi}{4}$$



$$w = \frac{z}{1-iz}$$



② Under the transformation $w = \frac{z-i}{z+2}$, find

the image of

(a) $\operatorname{Re}(z) = -1$

(b) $\operatorname{Im}(z) = 0$

Make z the subject :-

$$wz + 2w = z - i$$

$$2w + i = z - wz$$

$$2w + i = z(1-w)$$

$$z = \frac{2w+i}{1-w}$$

Let $z = x + yi$ and $w = u + vi$

$$x + yi = \frac{2(u+vi) + i}{1 - (u+vi)}$$

$$= \frac{2u + (2v+1)i}{(1-u) - vi} \times \frac{(1-u) + vi}{(1-u) + vi}$$

$$= \frac{2u(1-u) - v(2v+1) + i[2uv + (1-u)(2v+1)]}{(1-u)^2 + v^2}$$

$$x + yi = \frac{2u(1-u) - v(2v+1)}{(1-u)^2 + v^2} + i \left(\frac{2v + 1 - u}{(1-u)^2 + v^2} \right)$$

Now we can compare real and imaginary parts

(a)

$\text{Re}(z) = -1$

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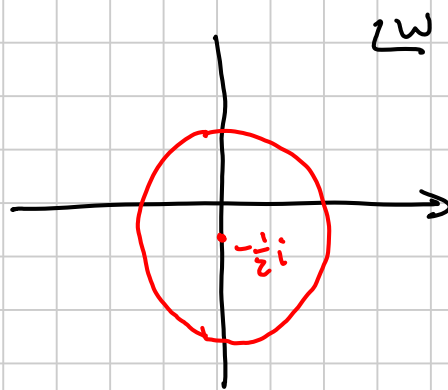
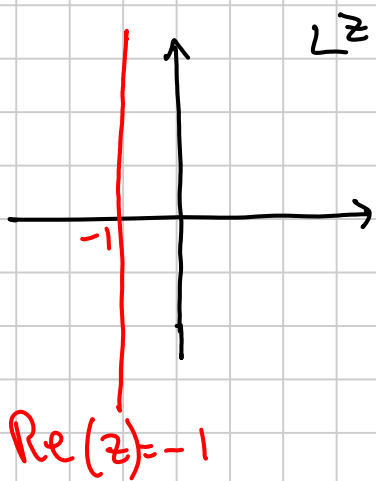
$$\Rightarrow \frac{2u - 2u^2 - 2v^2 - v}{(1-u)^2 + v^2} = -1$$

$$\Rightarrow 2u - 2u^2 - 2v^2 - v = -[1 - 2u + u^2 + v^2]$$

$$\Rightarrow 0 = u^2 + v^2 + v - 1$$

$$\Rightarrow u^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{4}$$

Circle, centre $(0, -\frac{1}{2})$ or $-\frac{1}{2}i$ radius $\frac{\sqrt{5}}{2}$



(b)

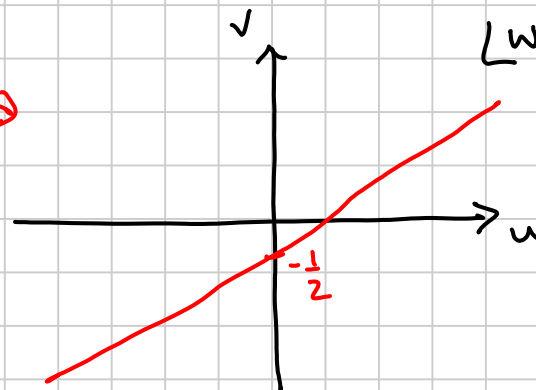
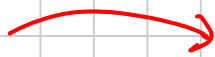
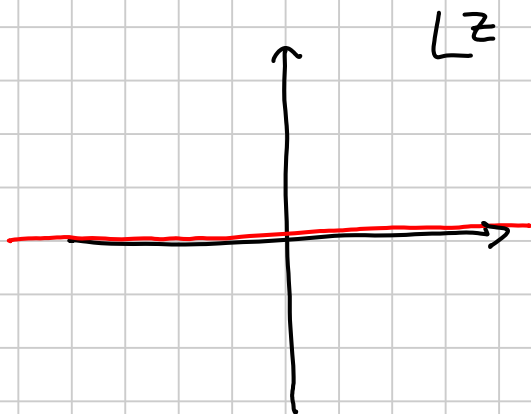
$\text{Im}(z) = 0$

$$\Rightarrow y = 0$$

$$\Rightarrow \frac{2v + 1 - u}{(1-u)^2 + v^2} = 0$$

$$\Rightarrow 2v + 1 - u = 0$$

$$\Rightarrow v = \frac{1}{2}u - \frac{1}{2}$$



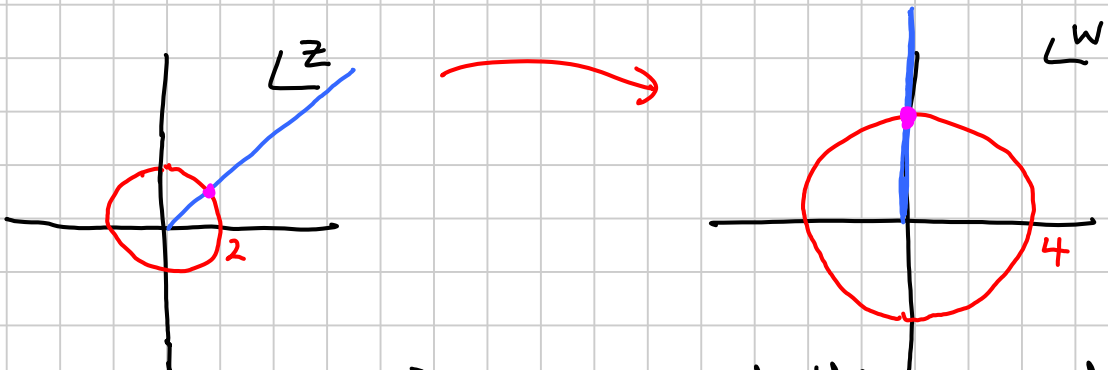
③ Under the transformation $w = z^2$, find the image of:

(a) $|z| = 2$ (b) $\arg(z) = \frac{\pi}{4}$ (c) $\operatorname{Re}(z) + \operatorname{Im}(z) = 1$

- For this transformation, making z the subject introduces square roots which makes for complications so we avoid doing this.

(a) $|w| = |z^2|$
 $= |z|^2$
 $= 4$

$|z z| = |z| |z|$



(but as the point z moves around the original circle, the point w moves around the image circle TWICE)

(b) $\arg(w) = \arg(z^2)$
 $= 2 \arg(z)$
 $= \frac{\pi}{2}$

(see above: image of blue line in z is the blue line in w)

(c) $\operatorname{Re}(z) + \operatorname{Im}(z) = 1$ ie/ $x + y = 1$

$$w = z^2$$

$$u + vi = (x + yi)^2$$

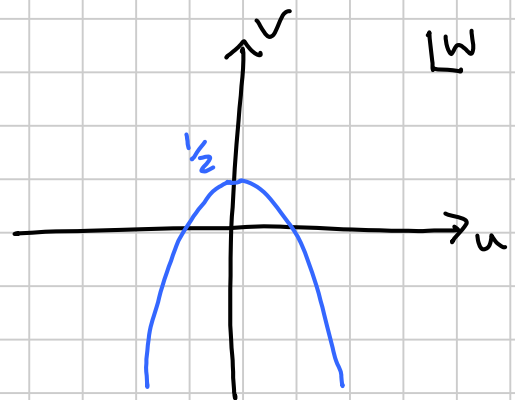
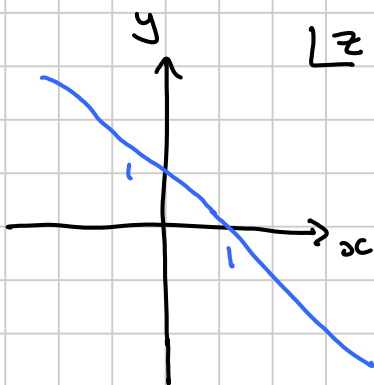
$$= (x^2 - y^2) + 2xyi$$

So $u = x^2 - y^2$
 $= (x+y)(x-y)$

But $x+y=1 \Rightarrow u = x-y$
 And $v = 2xy$

We need to eliminate x and y

$$\begin{aligned} u^2 &= x^2 - 2xy + y^2 \\ u^2 + v &= x^2 + y^2 - 2xy \\ &= (x+y)^2 - 2xy \\ &= \frac{1}{2}(x+y)^2 - v \\ u^2 + 2v &= 1 \\ v &= \frac{1}{2} - \frac{1}{2}u^2 \end{aligned}$$



p63 Ex 3.7 Q 1, 2a, 4, 5, 6, 7, 8, 9b, 10

The n^{th} roots of a complex number

Any number has n n^{th} roots.

Example Find the 5th roots of $w = 16 + 16\sqrt{3}i$

Write w in modulus-argument form:

$$\begin{aligned} |w| &= \sqrt{16^2 + (16\sqrt{3})^2} \\ &= \sqrt{16^2(1+3)} \\ &= 16 \times 2 = \underline{\underline{32}} \end{aligned}$$

$$\begin{aligned} \arg(w) &= \arctan\left(\frac{16\sqrt{3}}{16}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

So we need to find z such that

$$z^5 = 32e^{i\pi/3}$$

let $z = re^{i\theta} \Rightarrow r^5 e^{i5\theta} = 32e^{i\pi/3}$

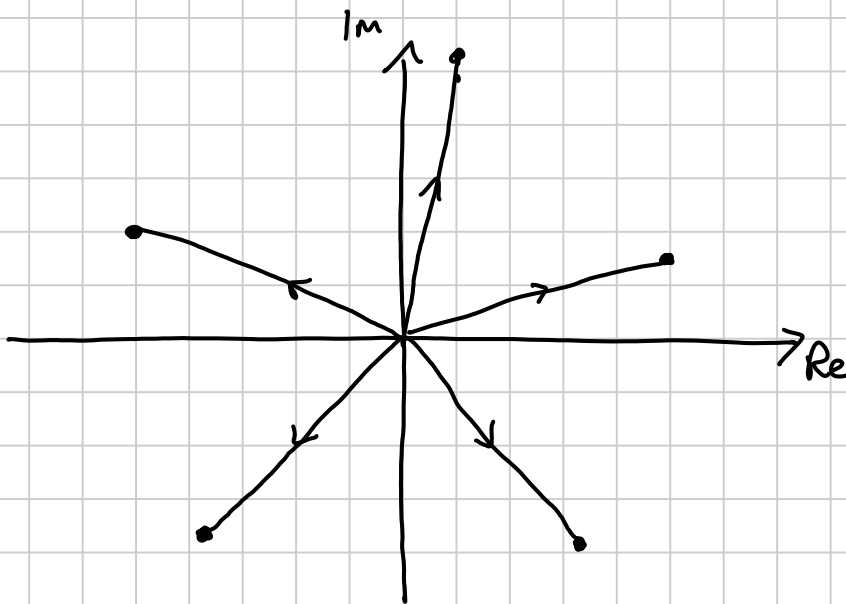
$$\Rightarrow r^5 = 32 \quad \text{so } r = 2$$

and $5\theta = \frac{\pi}{3} \text{ or } \frac{\pi}{3} + 2\pi \text{ or } \frac{\pi}{3} + 4\pi \text{ or } \frac{\pi}{3} - 2\pi \text{ or } \frac{\pi}{3} - 4\pi$

$$\theta = \frac{\pi}{15} \text{ or } \frac{7\pi}{15} \text{ or } \frac{13\pi}{15} \text{ or } -\frac{\pi}{3} \text{ or } -\frac{11\pi}{15}$$

$$z = 2e^{i\pi/15} \text{ or } 2e^{i7\pi/15} \text{ or } 2e^{i13\pi/15} \text{ or } 2e^{-i\pi/3} \text{ or } 2e^{-i11\pi/15}$$

On an Argand diagram :-



Note

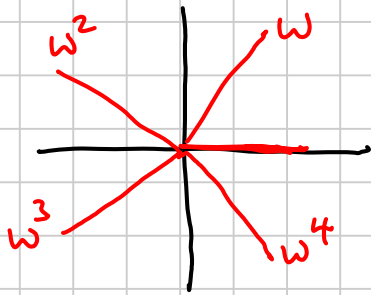
①

The n^{th} roots of any number, viewed as points on an Argand diagram, lie at the vertices of a regular n -gon. Viewed as vectors they form spokes at equally placed intervals

②

The sum of the n^{th} roots of any number is 0.

③ The n^{th} roots of 1 have some additional properties :—



One root is 1
If n is even, another root is -1
The rest of the roots occur in complex conjugate pairs.

The roots can be written as.

$$1, w, w^2, w^3, \dots, w^{n-1} \quad \text{where } w = e^{i\frac{2\pi}{n}}$$

p37 Ex 3.2 Q 1c, 2b, 3, 4, 5, 7ab, 8b, 10