COMPLEX NUMBERS

Note Title 21/06/2006

These are a natural extension of the number system so that we can find the square root of any number. They form a COMPLETE system – no further extension is necessary.

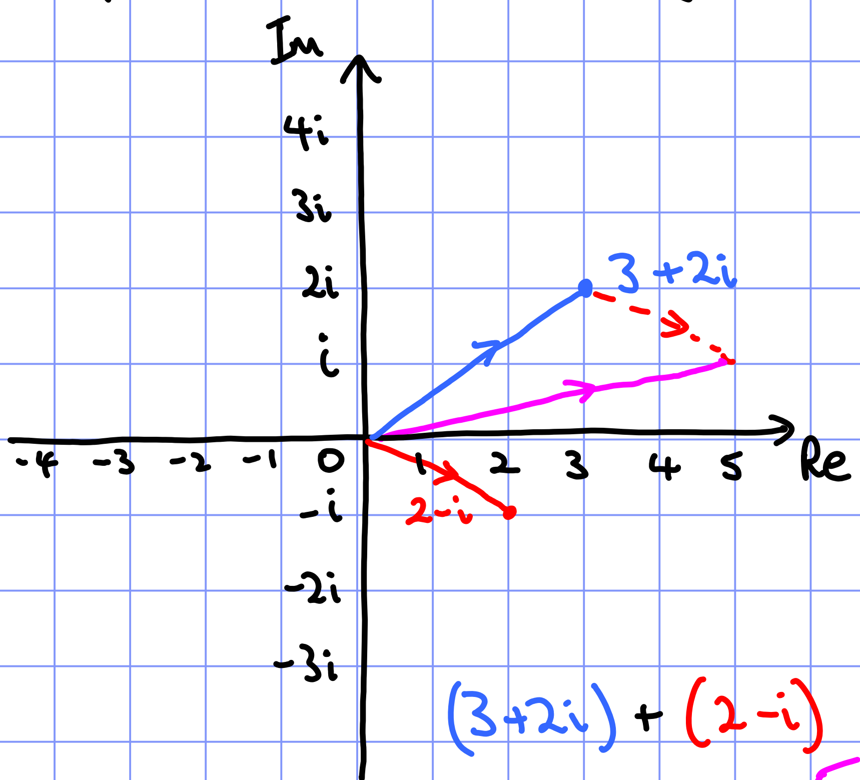
$$\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$$

We write $\sqrt{-1} = i$.

So $\sqrt{-4} = \sqrt{4 \times -1} = \sqrt{4} \times \sqrt{-1} = 2i$

$\sqrt{-2} = \sqrt{2 \times -1} = \sqrt{2} \times \sqrt{-1} = i\sqrt{2}$ etc.

Real numbers completely fill the number line; complex numbers are represented using a number <u>plane</u> called the 'Argand Diagram'



A number is represented either by a point, or by the vector from O to that point.

Adding and subtracting complex numbers corresponds to adding/subtracting their vectors

$(3+2i) + (2-i)$

$= 5+i$

Multiplying and dividing

① $(2+3i)(4-8i) = 8 + 12i - 16i - 24i^2$
$= \underline{\underline{32 - 4i}}$

② $i^3 = i^2 \times i$
$= -1 \times i$
$= \underline{\underline{-i}}$

$i^4 = \underline{\underline{1}}$ $i^5 = i^4 \times i$
$= \underline{\underline{i}}$ <u>etc</u>

The COMPLEX CONJUGATE of a complex number z $(= a+bi)$ is $z^* = a - bi$
Note that both $z + z^*$ and zz^* are always real numbers.

To <u>divide</u> two complex numbers we need to 'realize the denominator' by multiplying top and bottom by the complex conjugate.

③ $\dfrac{2+3i}{5-2i}$

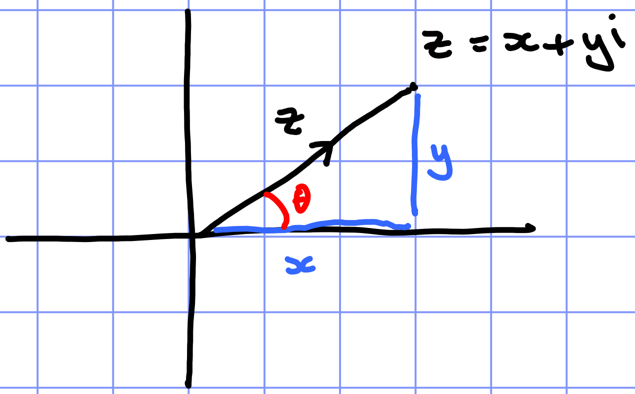
$= \dfrac{2+3i}{5-2i} \times \dfrac{5+2i}{5+2i}$

$= \dfrac{10 + 15i + 4i + 6i^2}{25 - \cancel{10i} + \cancel{10i} - 4i^2} = \dfrac{4 + 19i}{29}$

$= \underline{\underline{\dfrac{4}{29} + \dfrac{19}{29}i}}$

Modulus and Argument of a Complex Number

The modulus of a complex number is the length of its vector. It is written $|z|$

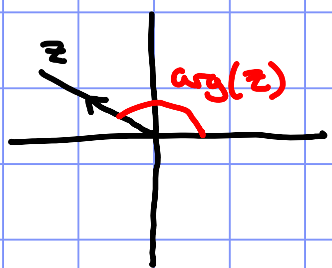


So $\boxed{|z| = \sqrt{x^2+y^2}}$

The argument of a complex number is the angle its vector makes with the positive x-axis (θ on the diagram above). It is written $\arg(z)$

So $\boxed{\arg(z) = \arctan\left(\frac{y}{x}\right) \text{ but take account of quadrant}}$

Example If $z = -\sqrt{3} + i$, find $\arg(z)$



$\arctan\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$

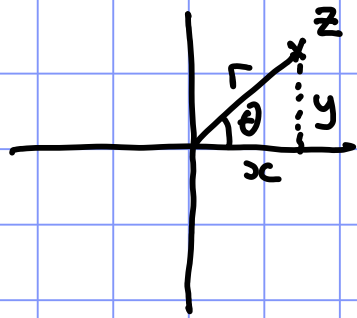
but $\arg(z) = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$

Writing numbers in modulus – argument form

The usual way of writing a complex number is $z = x + yi$

However, since

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$



we can write $z = r\cos\theta + ir\sin\theta$

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

r = modulus
θ = argument

which is the <u>modulus – argument form</u> of z.

This is useful for multiplying complex numbers:-

if
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

then
$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2\left[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right] \\ &= r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right] \end{aligned}$$

So $|z_1 z_2| = r_1 r_2$ and $\arg(z_1 z_2) = \theta_1 + \theta_2$

$$|z_1 z_2| = |z_1||z_2| \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Also
$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \times \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]}{r_2(\cos^2\theta_2 + \sin^2\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

So $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$ and $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$.

Summarizing,

$$|z_1 z_2| = |z_1||z_2| \qquad \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

(rules for arg are same as for log)

Example Simplify $\dfrac{6(\cos\frac{\pi}{3} + i\sin\pi/3)}{2(\cos\pi/6 - i\sin\pi/6)}$

$$= \frac{6(\cos\pi/3 + i\sin\pi/3)}{2(\cos(-\pi/6) + i\sin(-\pi/6))}$$

$$= 3\left(\cos\left(\pi/3 - \left(-\pi/6\right)\right) + i\sin\frac{\pi}{2}\right)$$

$$= \underline{\underline{3i}}$$

Equating Real and Imaginary Parts

If $z = x + yi$ and $w = u + vi$ and we know that $z = w$, then we often find it useful to split this into two equalities involving real numbers.

$$\left.\begin{array}{lll} Re(z) = Re(w) & \text{ie,} & x = u \\ Im(z) = Im(w) & \text{ie,} & y = v \end{array}\right\}$$

One application of this is in finding square roots.

<u>Example</u> Find $\sqrt{5+12i}$

Suppose $z^2 = 5 + 12i$

Write z as $x + yi$, so

$(x+yi)(x+yi)$ $\quad (x+yi)^2 = 5 + 12i$

$$\underline{(x^2 - y^2)} + \underline{2xyi} = \underline{5} + \underline{12i}$$

Equating real and imaginary parts

$$\left.\begin{array}{ll} ① & x^2 - y^2 = 5 \\ ② & 2xy = 12 \end{array}\right\}$$

$② \Rightarrow y = \dfrac{6}{x}$

Subst in ① $\Rightarrow x^2 - \dfrac{36}{x^2} = 5$

$$x^4 - 5x^2 - 36 = 0$$

$$(x^2 + 4)(x^2 - 9) = 0$$

$x^2 + 4 = 0$ has no solutions, since $x \in \mathbb{R}$.

$x^2 - 9 = 0 \Rightarrow \left.\begin{matrix} x = 3 \\ y = 2 \end{matrix}\right\}$ or $\left.\begin{matrix} x = -3 \\ y = -2 \end{matrix}\right\}$

So $z = 3 + 2i$ or $-3 - 2i$

ie/ $\underline{\underline{z = \pm (3 + 2i)}}$

Complex Roots of Polynomial Equations

The "Fundamental Theorem of Algebra" states that if $P(x)$ is a polynomial of degree n, ie/

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

(a_i are constants, $a_i \in \mathbb{C}$, $a_n \neq 0$)

then the equation $P(x) = 0$ has n solutions

$$\alpha_1, \alpha_2, \ldots \alpha_n$$

($\alpha_i \in \mathbb{C}$, some may be repeated roots)

It follows that $P(x)$ can be factorized into n linear factors

$$P(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$$

Example Solve the equation

$$x^2 + 2x + 5 = 0$$

and hence factorize $x^2 + 2x + 5$.

$$x = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$x = \underline{-1+2i} \text{ or } \underline{-1-2i}$$

So $x^2 + 2x + 5 = (x - (-1+2i))(x - (-1-2i))$
$= (x + 1 - 2i)(x + 1 + 2i)$

It can be shown that if all the coefficients of a polynomial $P(x)$ are real ($a_i \in \mathbb{R}$ above) then any complex roots of $P(x) = 0$ occur in complex conjugate pairs. (ie if α is one root, α^* is another root)
(It follows that any equation of ODD order must have at least one REAL root, and an ODD number of them.)

Example Solve the equation

$$x^4 - 6x^3 + 6x^2 + 34x - 195 = 0$$

given that one of the roots is $x = 2 - 3i$.

Another root is $2 + 3i$

So two of the factors are

$$\left(x - (2+3i)\right) \text{ and } \left(x - (2-3i)\right)$$

So a quadratic factor is

$$\left(x^2 - (2+3i)x - (2-3i)x + 13\right)$$

ie, $x^2 - 4x + 13$

$$
\begin{array}{r|l}
 & x^2 - 2x - 15 \\
\hline
x^2 - 4x + 13 & x^4 - 6x^3 + 6x^2 + 34x - 195 \\
 & \underline{x^4 - 4x^3 + 13x^2} \\
 & -2x^3 - 7x^2 + 34x \\
 & \underline{-2x^3 + 8x^2 - 26x} \\
 & -15x^2 + 60x - 195 \\
 & \underline{-15x^2 + 60x - 195} \\
 & 0
\end{array}
$$

So the other quadratic factor is $x^2 - 2x - 15$

$$= (x-5)(x+3)$$

So all 4 solutions are

$$\underline{\underline{x = 2+3i,\ 2-3i,\ 5 \text{ or } -3}}$$

and the factors are

$$(x-2-3i)(x-2+3i)(x-5)(x+3) = 0$$

(four linear factors)

or $$(x^2 - 4x + 13)(x-5)(x+3) = 0$$

(which is factorized as far as possible using only real numbers) p 39 Ex 2.6 Q 3 acegh, 8, 9

p 44 Ex 2.7 Q 3, 4, 5, 6, 7, 8, 12

p 48 Ex 2.8 Q 4,

p 51 Ex 2.9 Q 2, 5, 7