5 PROJECTILES

Objectives

After studying this chapter you should

- recognise that projectile motion is common;
- understand how to obtain a simple mathematical model of projectile motion;
- be able to validate the model;
- be able to solve simple problems of projectile motion;
- know how to use the model to investigate real life projectile problems.

5.0 Introduction

What do tennis and basket balls have in common with kangaroos?

The ball or body is in motion through the air, the only forces acting on it being its weight and the resistance to its motion due to the air. A motion like this is called a **projectile motion** and is very common especially in sport, for example basketball and tennis. The jumps of insects such as locusts, fleas and grasshoppers are projectile motions, as are the motions of a slate blown off a roof and a piece of mud or small stone thrown up from the road against a car windscreen. Road accidents often involve projectile motions, for example that of the shattered glass of a windscreen. The drops of water that form the jet from a hosepipe behave as projectiles. The Greeks and Romans used catapults to launch projectiles at their enemies, archers were important in medieval battles like Crécy and Agincourt, whilst guns have been a major weapon of war from the sixteenth century onwards.

Activity 1 Projectiles and sport

Make a list of sports which involve projectile motion. How many can you find?

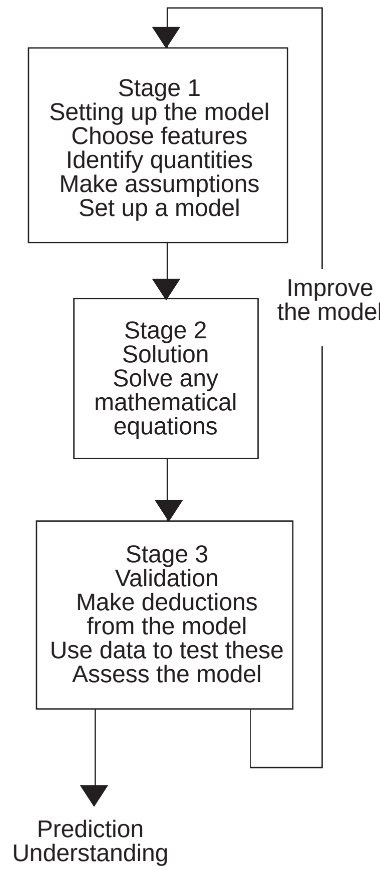
Make a list of non-sporting examples of projectile motion.

Why is it useful to investigate projectile motion?

Sports coaches want to know how to improve performance. Police accident investigators want to determine car speeds from the position of glass and other objects at the scene of an accident. In these and other instances mathematical modelling of projectile motion proves very useful.

5.1 Making a mathematical model

You met some examples of mathematical models in Chapter 1, where they take the form of a graph or equation obtained from data collected in experiments. In Chapter 2 you were introduced to Newton's laws; the models in this chapter use these laws to attempt to answer interesting questions about everyday situations. How can I improve my performance at basketball? Why do the chairs swing out on the chair-o-plane ride at a theme park? To shed light on these questions with a mathematical model it is necessary to decide what are the important quantities - e.g. speed, the height of the basket, the weight of the riders in the sport or ride - and to make some assumptions or informed guesses, as to the relations between them.. The resulting model is usually a set of equations. Their solution gives results which need to be tested out against the original situation to see that they make sense. This is called **validating** the model. A model which agrees well with the real situation can be used to make **predictions** about it. One which does not agree well needs modifying, for example, by asking if all the important quantities really have been taken into account.



The diagram opposite summarises this modelling process.

This diagram is basic to what follows. You should use it in any modelling exercises you do.

5.2 Setting up a model for projectile motion

Choosing features and identifying quantities

When a projectile such as a basketball is thrown, it describes a path through the air, first ascending and then descending but also travelling forwards. To describe the motion a mathematical model needs to give the position and velocity of the projectile at any point of its path as functions of time and reproduce the features just described.

Activity 2 What determines the motion of a projectile?

Make a list of the quantities you think determine the motion of a projectile such as a basketball.

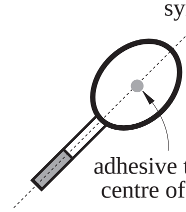
Which do you think are the most important?

You may have on your list the size and shape of the projectile. The next activity explores how important these are.

Activity 3 The motion of a tennis racket

You need a tennis racket (or similar) and a piece of coloured adhesive tape for this activity.

Find the position of the centre of mass of the tennis racket. One way to do this is to find the point about which the racket balances on your finger.



Stick the piece of tape at the centre of mass.

Have two friends throw the racket between them and watch the motion of the tape.

Is the motion of the tape similar to the projectile motion of a ball?

Making assumptions

Activity 3 suggests that, even when the tennis racket rotates, the motion of the centre of mass is as if there were a particle there. This suggests

Assumption 1 The projectile is treated as a particle.

This is reasonable for a tennis or basket ball provided the ball does not have too much spin or rotation. Large amounts of spin on a ball can, however, significantly affect the motion. Treating the ball as a particle leaves out spin and rotation. Even replacing a kangaroo by a particle at its centre of mass gives results which tally with what happens in the hops of real kangaroos.

One way in which spin affects the motion of a ball is by deflecting the motion out of a vertical plane. Side winds can have the same effect. Many projectiles do, however, move in more or less the one vertical plane and these, being the simplest, are the ones modelled here.

Assumption 2 The motion is in one vertical plane.

You may have the resistance of the air to a projectile's motion on your Activity 2 list. However, since many projectiles are in flight for quite short times, air resistance is probably not very important in their motion compared to gravity.

Assumption 3 Air resistance is negligible.

For a model based on these assumptions the motion of the projectile depends at most on

its initial speed U and angle of projection θ;

its point of release;

its mass m;

gravity.

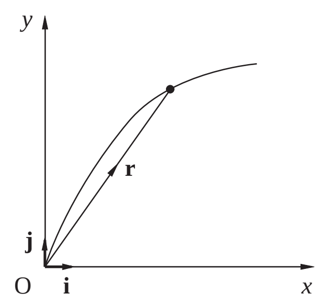
Are these the quantities you decided were most important in Activity 2?

Setting up the model

Projectile motion must obey Newton's Second Law

$$\text{force} = \text{mass} \times \text{acceleration} .$$

Since air resistance is assumed negligible, the only force on the projectile is its weight which acts vertically downward at every point of the path.



To give the position of the projectile at any time t some coordinates are needed. A convenient origin is the point O at which the projectile is released and the time at which this happens is taken as $t = 0$. Cartesian axes Ox, Oy, are then chosen along the horizontal and vertical through O.

The **position** of the projectile at any time t is given by its position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} ,$$

where **i** and **j** are unit vectors along Ox and Oy. Since **r** varies with t, x and y are functions of t, which it is the aim of the model to determine.

The **velocity v** of the projectile is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j},$$

whilst its **acceleration a** is

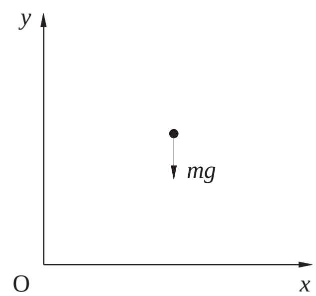
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}.$$

The basic equation of the model

The force on the projectile at time t is

$$\mathbf{F} = 0\mathbf{i} - mg\,\mathbf{j}.$$

Newton's Second Law then gives $m\mathbf{a} = \mathbf{F}$ or



$$m\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - mg\,\mathbf{j}.$$

Dividing through by m gives

$$\frac{d^2\mathbf{r}}{dt^2} = 0\mathbf{i} - g\,\mathbf{j}. \qquad (1)$$

This is the basic equation of the model.

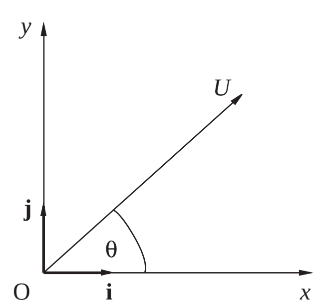
Deduction 1

Since this equation does not involve m, the motion is independent of the mass of the projectile.

5.3 Solving the basic equation of the model

Integrating equation (1) gives

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = A\mathbf{i} + (B - gt)\mathbf{j} \qquad (2)$$



where A and B are constants.

Since the projectile is released from O at time $t = 0$ with speed U at an angle θ to the horizontal, its velocity at $t = 0$ is

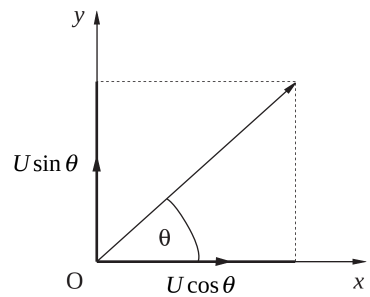
$$\mathbf{v} = U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j}.$$

Using this and substituting $t = 0$ in equation (2) gives

$$U\cos\theta\,\mathbf{i} + U\sin\theta\,\mathbf{j} = A\mathbf{i} + (B - g0)\mathbf{j},$$

so that

$$A = U\cos\theta, \qquad B = U\sin\theta.$$



The velocity at time t is then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = U\cos\theta\,\mathbf{i} + (U\sin\theta - gt)\,\mathbf{j}. \qquad (3)$$

Integrating this equation gives

$$\mathbf{r} = (Ut\cos\theta + C)\,\mathbf{i} + \left(Ut\sin\theta - \tfrac{1}{2}gt^2 + D\right)\mathbf{j}.$$

where C and D are constants. When $t = 0$, $\mathbf{r} = C\mathbf{i} + D\mathbf{j}$, and, since $\mathbf{r} = 0$ when $t = 0$, $C = D = 0$. This gives

$$\mathbf{r} = Ut\cos\theta\,\mathbf{i} + \left(Ut\sin\theta - \tfrac{1}{2}gt^2\right)\mathbf{j} \qquad (4)$$

Taking components of equations (3) and (4) gives the basic results of the model for projectile motion.

Deduction 1

$$\frac{dx}{dt} = U\cos\theta \qquad (5)$$

$$\frac{dy}{dt} = U\sin\theta - gt \qquad (6)$$

$$x = Ut\cos\theta \qquad (7)$$

$$y = Ut\sin\theta - \tfrac{1}{2}gt^2. \qquad (8)$$

These equations give the position and velocity of the projectile at any time. Do they remind you of any equations you have already met? If so, why do you think this should be?

Equation (5) shows that the horizontal component of velocity, $\frac{dx}{dt}$, is constant throughout the motion.

Could you have predicted this?

Activity 4 A first exploration of the model

You need a calculator for this activity. Take $g = 10\ \text{ms}^{-2}$.

A tennis ball is given an initial speed of 30 ms^{-1}.

Find the horizontal and vertical components of the velocity of the ball for an angle of projection of $30°$ to the horizontal at times in seconds 0, 0.5, 1.0, 1.5, 2.0, 2.5, 3.0. Find also x and y at these times.

At what time is the vertical component of velocity zero? How high has the ball then risen and how far has it travelled horizontally?

When is the ball again on the same level as its point of projection? What is the distance R it has then travelled horizontally and what are its velocity components?

Repeat for angles of projection of $15°$, $45°$, $60°$ and $75°$.

For which of these angles of projection does the ball travel furthest? What do the results you obtain for the distance R for the various angles of projection suggest?

Example

A ball is thrown with a speed of 8 ms^{-1} at an angle of 30° to the horizontal. How high above its point of projection is it when it has travelled 2 m horizontally?

Solution

The time for the ball to travel 2 m horizontally is given by equation (7), $x = Ut\cos\theta$, as

$$t = \frac{x}{U\cos\theta},$$

Here $x = 2$, $U = 8$, $\theta = 30$, so that

$$t = \frac{2}{8\cos 30} = 0.29\,\text{s}$$

Taking $g = 10\ \text{ms}^{-2}$,

the height of the ball above the ground at this time t is given by equation (8) as

$$y = U\sin\theta t - \tfrac{1}{2}gt^2$$

$$= 8\sin 30 \times 0.29 - 5\times(0.29)^2$$

$$\approx 0.74 \text{ m}$$

Hence height is 0.74 m.

Exercise 5A

1. A ball is thrown with initial speed 20 ms^{-1} at an angle of $60°$ to the horizontal. How high does it rise? How far has it then travelled horizontally?

2. A ball is kicked with speed 25 ms^{-1} at an angle of projection of $45°$. How high above the ground is it when it has travelled 10 m horizontally?

3. An arrow is fired from a bow with a speed of $50\,\text{ms}^{-1}$ at an angle of $5°$ to the horizontal. What is its speed and the angle its velocity makes with the horizontal after 0.6 s?

4. A stone is thrown with speed 10 ms^{-1} at an angle of projection of $30°$ from the top of a cliff and hits the sea 2.5 s later. How high is the cliff? How far from the base of the cliff does the stone hit the water?

5.4 Validating the model

Activity 4 suggests that the model is reproducing at least some of the main features of projectile motion in a special case. The projectile ascends then descends and at the same time travels forward. A more testing validation is to deduce what path the model says the projectile follows and then test this experimentally.

The path of the projectile

Activity 5 Plotting the path

You will need graph paper and a calculator or a graphic calculator for this activity.

(a) For $U = 10\text{ ms}^{-1}$, $\theta = 30°$, find the values of x and y at time intervals of 0.1 s from equations (7) and (8).

(b) Plot the points (x, y) to obtain the path of the projectile.

(c) Repeat for $U = 10\text{ ms}^{-1}$, $\theta = 60°$.

You should find that both paths are symmetrical about the vertical through their highest points.

You will need your plot again in Activity 9.

The path of a projectile is an example of a curve called a **parabola**.

The equation of the path for general values of U and θ is found by eliminating the time t between equations (7), $x = Ut\cos\theta$ and (8), $y = Ut\sin\theta - \frac{1}{2}gt^2$.

From equation (7)

$$t = \frac{x}{U\cos\theta}.$$

Substituting this into equation (8) gives

$$y = U\sin\theta\left(\frac{x}{U\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{U\cos\theta}\right)^2$$

Remember $\sec\theta = \dfrac{1}{\cos\theta}$

or

$$y = x\tan\theta - g\frac{\sec^2\theta}{2U^2}x^2$$

Deduction 2

The path of the projectile is the parabola

$$y = x\tan\theta - g\frac{\sec^2\theta}{2U^2}x^2. \qquad (9)$$

Example

A jet of water flows from a hosepipe with speed 40 ms^{-1} at an angle of $60°$ to the horizontal. Given that the particles of water travel as projectiles, find the equation of the path of the jet.

Solution

The equation of the path can be found by substituting $U = 40$, $\theta = 60$, in equations (7) and (8) to give

$$x = 40\cos 60°\, t, \quad y = 40\sin 60°\, t - \tfrac{1}{2}gt^2,$$

or $$x = 20t, \qquad y = 20\sqrt{3}\,t - 5t^2.$$

From the first of these equations $t = \dfrac{x}{20}$.

Substituting this into the second equation gives

$$y = 20\sqrt{3}\left(\frac{x}{20}\right) - 5\left(\frac{x}{20}\right)^2$$

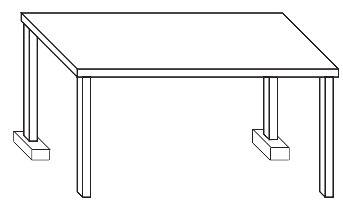
or

$$y = 1.73x - 0.0125x^2.$$

Testing this deduction experimentally

Activity 6 Finding the path of a ball

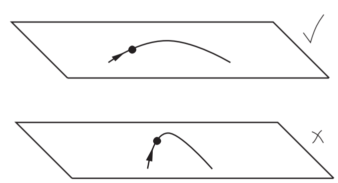
You need a table, two equal blocks, a billiard or squash ball, a marker pen, sugar paper, Blu-tak, water and squared paper for this activity.



Use the blocks to give the table a gentle incline.

Blu-tack the sugar paper to the table.

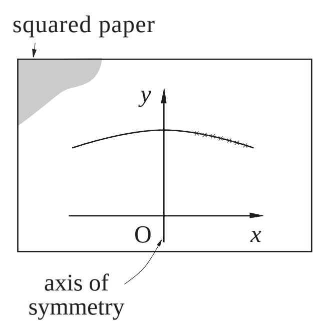
Practise releasing the ball so that its path lies well on the paper. A reasonably shallow trajectory as shown in the diagram usually gives the best results.



Wet the ball and release it.

Mark the track before it dries and cut it out.

Put the track on squared paper and insert axes.



Choose some points on the path and write down their coordinates.

Find the equation of the path that goes through these points. You can do this with a graphic calculator or a function graph plotter program.

As it is difficult to plot the path of a ball moving through the air, a table is used in Activity 6.

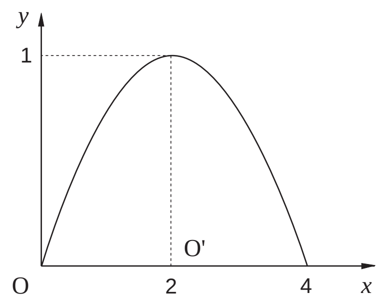
The acceleration of the rolling ball is down the table but has a value less than g. The same model as for motion under gravity can be used for the motion of the ball on the table with the axes Ox and Oy along and up the line of greatest slope of the table and g replaced by a smaller value than 10 ms^{-2}. In particular, the model predicts the path of the ball is a parabola.

Is this what you find?

The equation you are likely to find is not $y = ax - bx^2$, the form of equation (9). This is because the origin and y-axis are different. It is easier to fit an equation to the path when its axis of symmetry is the y-axis. An example shows how to go from one equation to the other.

Example

For $U = 2\sqrt{10}$ ms^{-1}, and $\theta = 45°$, the path of the projectile from equation (9) is



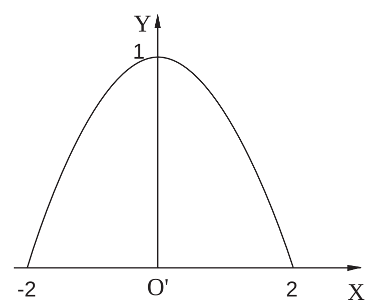
$$y = -\tfrac{1}{4}x^2 + x \ .$$

Completing the square on the right gives

$$y = -\tfrac{1}{4}\left(x^2 - 4x\right)$$

$$= -\tfrac{1}{4}\left[(x-2)^2 - 4\right],$$

so $$y = 1 - \tfrac{1}{4}(x-2)^2 .$$



Setting $\mathrm{X} = x - 2, \quad \mathrm{Y} = y$, gives

$$\mathrm{Y} = 1 - \tfrac{1}{4}\mathrm{X}^2$$

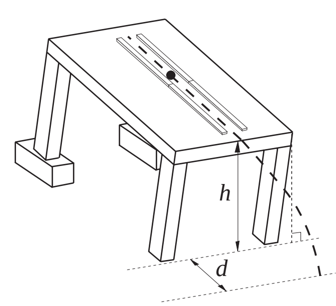
referred to the axes shown in the diagram.

You should obtain the equation for the trace of the path in this form, that is, $\mathrm{Y} = c - d\mathrm{X}^2$.

*Activity 7 Finding the path of a ball from photographs

If it is possible to obtain time lapse photos of the motion of a ball, then these could be made the basis of the validation of the equation of the path but now with the motion under gravity.

Activity 8 Another validation you may like to try



You will need a table, two equal blocks, four metre rulers, a squash ball, Blu-tack, a marker pen and paper for this activity.

Use the blocks to give the table a gentle incline longways.

Make a track along the table with the metre rulers as in Galileo's experiment in Chapter 2.

Again, as in Galileo's experiment, find the speed of the ball at the end of the track.

Record this speed.

Find where the ball lands on the floor.

Model the ball as a projectile and from measurements of h and d calculate its speed of projection on leaving the track.

How does this compare with the speed you found at the end of the track?

The velocity of the projectile

The speed of the projectile at time t is the magnitude $|\mathbf{v}|$ of its velocity $\mathbf{v}$, given by

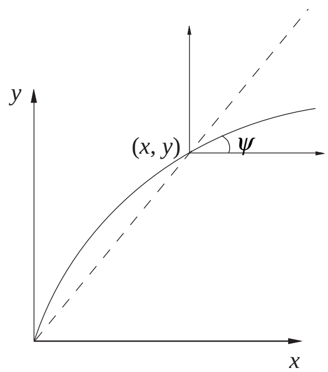
$$|\mathbf{v}| = \left|\frac{d\mathbf{r}}{dt}\right| = \left|\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right|$$

and, since $|a\mathbf{i} + b\mathbf{j}| = \sqrt{a^2 + b^2}$, this can be written as

$$|\mathbf{v}| = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}}.$$

The angle ψ the velocity makes with the horizontal is given by

$$\tan\psi = \left(\frac{dy}{dt}\right) \Big/ \left(\frac{dx}{dt}\right).$$



Activity 9 The direction of the projectile's velocity

For $U = 10\ \text{ms}^{-1}$, $\theta = 30°$, find the values of the velocity components $\frac{dx}{dt}$ and $\frac{dy}{dt}$ at time intervals of 0.5 s from equations (5) and (6).

Calculate the angle ψ the velocity makes with the horizontal at the points corresponding to these times.

On the plot of the path you drew in Activity 5 indicate the direction of the velocity at each of these points.

You should confirm that the velocity is along the tangent to the path.

Do you think that your results from Activities 6 to 9 validate the projectile motion model? How might the model be improved?

Example

A stunt motorcyclist takes off at a speed of 35 ms^{-1} up a ramp of 30° to the horizontal to clear a river 50 m wide. Does the cyclist succeed in doing this?

Solution

The cyclist clears the river if the horizontal distance travelled is greater than 50 m.

To find this distance it is necessary to first find the time of flight by putting $U = 35$, $\theta = 30°$, $y = 0$, in equation (8) with $g = 10 \text{ ms}^{-2}$.

This gives

$$0 = 35 \sin 30° t - 5t^2 .$$

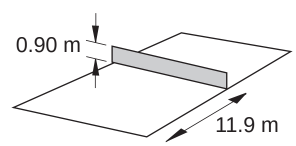
This equation has two solutions $t = 0$ and $t = 7 \sin 30° = 3.5$ s. The solution $t = 0$ gives the cyclist's take-off time so the time of flight is $t = 3.5$ s.

Substituting $U = 35$, $\theta = 30°$, $t = 3.5$, in equation (7) gives the horizontal distance x travelled as

$$x = 35 \cos 30° \times 3.5 = 106 \text{ m}.$$

So the cyclist easily clears the river.

Example



A tennis player plays a ball with speed 20 ms^{-1} horizontally straight down the court from the backline. What is the least height at which she can play the ball to clear the net? How far behind the net does the ball land when it is played at this height?

Solution

The time for the ball to reach the net is given by equation (7) on substituting $x = 11.90$, $U = 20$, $\theta = 0$, as

$$t = \frac{x}{U\cos\theta} = \frac{11.9}{20} = 0.595 \text{ s}.$$

The distance y below the point of play of the ball after this time is given by equation (8) on substituting $U = 20$, $\theta = 0$, $t = 0.595$ s, $g = 10$ ms^{-2} as

$$\begin{aligned} y &= (20\sin\theta)(0.595) - 5(0.595)^2 \\ &= -1.77. \end{aligned}$$

To clear the net the ball must be played from a height of at least $1.77 + 0.90 = 2.67$ m.

When played from this height, the ball hits the court when $y = -2.67$. The time t when this happens comes from equation (8) by substituting $y = -2.67$, $U = 20$, $\theta = 0$, so that

$$-2.67 = -5t^2$$

which gives $t = 0.73$ s.

From equation (7) the horizontal distance x travelled in this time is

$$x = 20 \times 0.73 = 14.56 \text{ m} .$$

The ball then lands $(14.56 - 11.90) = 2.66$ m behind the net.

Exercise 5B

1. David kicks a ball with a speed of 20 ms^{-1} at an angle of $30°$ to the horizontal. How far away from him does the ball land?
2. In the Pony Club gymkhana Carol wants to release a ball to drop into a box. The height above the box from which she drops the ball is 1.5 m and the pony's speed is 12 ms^{-1}. How far from the box should Carol drop the ball?
3. A bowler releases a cricket ball from a height of 2.25 m above the ground so that initially its path is level. Find the speed of delivery if it is to hit the ground a horizontal distance of 16 m from the point of release.
4. Karen is standing 4 m away from a wall which is 2.5 m high. She throws a ball at 10 ms^{-1} at an angle of $40°$ to the horizontal at a height of 1 m above the ground. Will the ball pass over the wall?
5. A bushbaby makes hops with a take-off speed of 6 ms^{-1} and angle of $30°$. How far does it go in each hop?
6. A stone is thrown up at an angle of $30°$ to the horizontal with a speed of 20 ms^{-1} from the edge of a cliff 15 m above sea level so that the stone lands in the sea. Find how long the stone is in the air and how far from the base of the cliff it lands. What are the speed and direction of the stone as it hits the water?
7. A ball is thrown with speed U at an angle of projection of $30°$. Show that at time t its speed q is given by

   $$q^2 = U^2 - Ugt + g^2t^2$$

   Find the height y to which it has risen in this time and hence show that

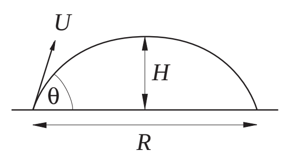
   $$q^2 = U^2 - 2gy\ .$$

   Do you think this result holds whatever the angle of projection?

5.5 More deductions from the model

Activity 4 explores how high the projectile rises and how far it travels horizontally in particular cases. These questions can, however, be answered generally.

How high does a projectile rise?



A projectile reaches its maximum height H when its vertical velocity component $\frac{dy}{dt} = 0$. The vertical velocity component at time t is given by equation (6)

$$\frac{dy}{dt} = U\sin\theta - gt.$$

Let $\frac{dy}{dt} = 0$ then $0 = U\sin\theta - gt$

which gives

$$t = (U\sin\theta)/g.$$

Putting $y = H$ and $t = \dfrac{U\sin\theta}{g}$ in equation (8) gives

$$H = U\sin\theta.\left(\frac{U\sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{U\sin\theta}{g}\right)^2,$$

so that

$$H = \frac{U^2\sin^2\theta}{2g}.$$

This is the same height as that to which a ball thrown vertically upwards with speed $U\sin\theta$ rises.

Deduction 3

The maximum height the projectile reaches above the point of release is

$$H = \frac{U^2\sin^2\theta}{2g}.$$

How far does a projectile travel horizontally?

The horizontal distance the projectile has travelled when it is again on Ox is called the **range** and is denoted by R.

The total time of flight T is obtained by putting $y = 0$ in equation (8), giving

$$0 = (U\sin\theta)t - \frac{1}{2}gt^2$$

or

$$0 = t\left(U\sin\theta - \frac{1}{2}gt\right).$$

This equation has two solutions, $t = 0$ and $t = \dfrac{2U\sin\theta}{g}$. The solution $t = 0$ is the time of projection from the origin 0 so that $t = \dfrac{2U\sin\theta}{g}$ is the time of flight T.

Deduction 4

The time of flight of the projectile is

$$T = \frac{2U\sin\theta}{g}.$$

This shows the total time of flight is twice the time to the maximum height.

The horizontal distance travelled at time t is $x = Ut\cos\theta$.

From this equation, the range R is

$$R = (U\cos\theta)T = 2U^2\frac{\sin\theta\cos\theta}{g} = \frac{U^2\sin 2\theta}{g}$$

(Remember $\sin 2\theta = 2\sin\theta\cos\theta$)

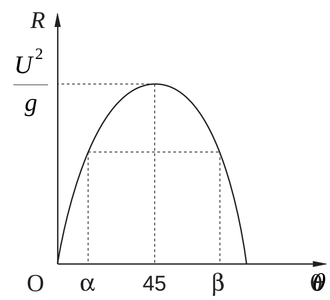
Deduction 5

The range of the projectile is

$$R = \frac{U^2\sin 2\theta}{g}.$$

For a given speed of projection U the range R is a function of θ. It is a maximum, R_{max}, when $\sin 2\theta = 1$, that is $2\theta = 90°$ or $\theta = 45°$.

$$R_{max} = \frac{U^2}{g}$$



graph range R against launch angle θ

The graph of R against θ shows that for values of R less than R_{max} there are two angles of projection which give the same angle. One of these, $\theta = \alpha$, is less than $45°$ and the other, $\theta = \beta$, greater than $45°$. Since the graph is symmetrical about the line $\theta = 45°$,

$$\beta = 90 - \alpha$$

Is this what you found in Activity 4?

Activity 10 Maximum ranges in sports

Some typical initial speeds of projectiles in sports are

golf ball	70 ms^{-1}
long jumper	10 ms^{-1}
rugby and soccer balls	30 ms^{-1}
table tennis ball	25 ms^{-1}
tennis ball	40 ms^{-1}
water polo ball	15 ms^{-1}

Find the maximum range for each of these from the projectile model.

In practice not all the projectiles in Activity 10 attain their maximum range. Why do you think this might be?

Example

A projectile, given an initial speed of 20 ms^{-1}, travels a horizontal distance 30 m. What are its possible angles of projection?

Solution

The time of flight is first found from equation (8); equation (7) then gives the angle of projection θ.

Putting $y = 0$, $U = 20$, in equation (8) gives

$$0 = 20\sin\theta t - 5t^2,$$

which has solutions $t = 0$ and $t = 4\sin\theta$. The time of flight is then $t = 4\sin\theta$. Substituting $x = 30$, $U = 20$, $t = 4\sin\theta$, in the equation for x gives

$$30 = (20\cos\theta)(4\sin\theta)$$

or

$$0.75 = 2\sin\theta\cos\theta$$

so that

$$\sin 2\theta = 0.75$$

and hence $\theta = 24.3°$ or $65.7°$.

Example

A tennis player makes a return at a speed of 15 ms^{-1} and at a height of 3 m to land in the court at a horizontal distance of 12 m from her. What are the possible angles of projection of the ball?

Solution

Let the ball travel a time t before hitting the court, θ the angle of projection.

The horizontal distance, $x = 12$, travelled in this time is given by $x = Ut\cos\theta$ with $U = 15$ as

$$12 = 15t\cos\theta\ ,$$

so that

$$t = \frac{4}{5\cos\theta}.$$

When the ball hits the court, $y = -3$, so the equation of the path gives

$$-3 = 15\sin\theta\, t - 5t^2.$$

Substituting for t gives

$$-3 = 12\tan\theta - 5\left(\frac{4}{5\cos\theta}\right)^2 ,$$

Remember $\sec\theta = \dfrac{1}{\cos\theta}$

or

$$-3 = 12\tan\theta - \frac{16}{5}\sec^2\theta .$$

This gives

$$\frac{16}{5}\left(1+\tan^2\theta\right) - 12\tan\theta - 3 = 0$$

Remember $\sec^2\theta = 1+\tan^2\theta$

or

$$\tan^2\theta - \frac{15}{4}\tan\theta + \frac{1}{16} = 0 .$$

Solving this quadratic in $\tan\theta$ gives

$$\tan\theta = \frac{\frac{15}{4} \pm \sqrt{\frac{221}{16}}}{2} = \frac{15 \pm \sqrt{221}}{8}$$

which gives $\theta = 1.0°$ or $75.0°$.

The most likely angle of projection is $1.0°$ since the other angle would give the player's opponent great scope for return.

Exercise 5C

1. A ball is thrown with a speed of 12 ms^{-1} at an angle of $30°$ to the horizontal. Find the maximum height to which it rises, the time of flight and the range. Find also the speed and direction of flight of the ball after 0.5 s and 1.0 s.

2. The initial speed of a projectile is 20 ms^{-1}. Find the two angles of projection which give a range of 30 m and the times of flight for each of these angles. What is the maximum range that can be achieved?

3. A projectile has range 100 m and reaches a maximum height of 20 m. What is its initial speed and angle of projection?

4. What is the least speed of projection with which a projectile can achieve a range of 90 m? What is the time of flight for this speed?

5. Robin Hood shoots an arrow with a speed of 60 ms^{-1} to hit a mark on a tree 60 m from him and at the same level as the arrow is released from. What are his possible angles of projection and which one is he likely to choose?

6. A locust can make long jumps of 0.7 m at a take-off angle of $55°$. Use the projectile model to find its take-off speed and the maximum height it reaches. (The take-off speed of locusts is observed to be about 3.4 ms^{-1}, which is higher than the value found using the projectile model. Why do you think this should be?).

7. A ball is thrown so that it goes as high as it goes forward. At what angle is it thrown?

8. A ball is thrown from a point O with speed 10 ms^{-1} at an angle θ to the horizontal. Show that, if it returns to the ground again at a distance from O greater than 5 m, then θ lies between $15°$ and $75°$ whilst the time of flight is between 0.52 s and 1.93 s.

5.6 Using the model

You may like to try some of the following investigations which give you a chance to use the projectile model in some real situations.

Activity 11 Accident!

Police Accident Investigation Units use the projectile model to estimate speeds of vehicles involved in accidents as in this hypothetical case.

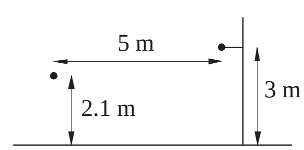
A builder's van collides with a low stone wall. Several lengths of timber fastened longitudinally to the roof of the van 1.8 m above the ground are projected forwards over the wall to land sticking in a muddy patch of field 9 m in front of the roof. At the same time the windscreen shatters and the bulk of the glass is found in the field beyond the wall between 23 m and 27 m in front of the windscreen.

The Accident Investigation Unit want the likely range of speeds for the car when it struck the wall. Use the projectile model to obtain estimates for this range. You should take into account that these estimates may need to be defended in court.

What might affect the reliability of your estimates?

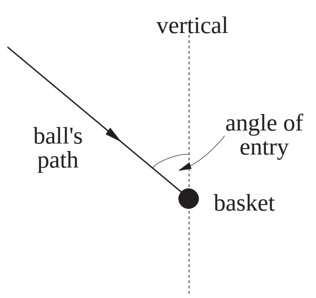
Activity 12 Making a basketball shot

Susan wants to make a clean shot, one that passes through the basket without hitting the rim or bouncing off the backboard. She can give the ball a speed of projection of 8 ms^{-1}.



Treat both the basket and the ball as points.

Investigate the angle of projection that puts the ball into the basket and the corresponding angle of entry.



What angle of projection should Susan aim for?

In reality the basket has a diameter of 0.45 m and the ball a diameter of 0.24 m.

Investigate how much margin Susan has on her preferred angle of projection.

Activity 13 How to improve at shot putting

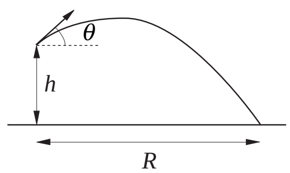
Coaches often concentrate on improving the speed at which a putter projects the shot rather than the angle. Why?

Choose a value for θ, say $\theta = 45°$, the angle for maximum range and a value for U, a reasonable one being $U = 12 \text{ ms}^{-1}$. (The height from which the shot is projected is ignored for simplicity)

Investigate the percentage changes in the range R for changes in U, say 5, 10, 15, 20% increases.

For a given U, again, say $U = 12 \text{ ms}^{-1}$, choose values of θ less and greater than 45°, say $\theta = 42°$ and $\theta = 48°$, and find the percentage changes in R for changes in U about these values.

What conclusions do you reach?



A putter releases the shot at some height above the ground. The range R is measured along the ground.

Investigate numerically how R varies with different heights h of projection and what angles give the maximum range for different h. Typical values of h are around 2 m.

Does a tall shot putter have an advantage over a short one?

5.7 Miscellaneous Exercises

1. Rashid is standing 4 m away from a wall which is 5 m high. He throws a ball at 10 ms^{-1} at an elevation of 40° above the horizontal and at a height of 1 m above the ground. Will the ball pass over the wall?

   If he throws at an angle θ, show that θ must satisfy

   $$\tan^{-1} 2 < \theta < \tan^{-1} 3$$

   for the ball to clear the wall.

2. A particle is projected with speed U at an angle θ to the horizontal. Show that its velocity at the point (x, y) on its path makes an angle ψ with the horizontal, where

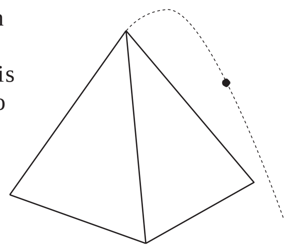
   $$\tan\psi = \frac{2y}{x} - \tan\theta .$$

   For $U = 10 \text{ ms}^{-1}$, $\theta = 45°$, show that

   $$y = x - 0.1x^2 .$$

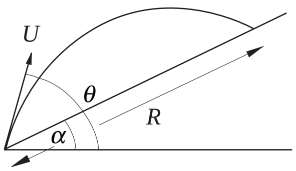
   Hence graph ψ against x for x at intervals of 0.5 m. What do you deduce about ψ?

3. One of the Egyptian pyramids is 130 m high and the length of each side of its square base is 250 m. Is it possible to throw a stone with initial speed 25 ms^{-1} from the top of the pyramid so that it strikes the ground beyond the base?



4. A ball is projected with speed U at an angle θ to the horizontal up the line of greatest slope of a plane inclined at an angle $\alpha(\alpha < \theta)$ to the horizontal. The ball strikes the plane again at a distance R from the point of projection. Show that

$$R = \frac{2U^2 \sin(\theta - \alpha)\cos\theta}{g\cos^2\alpha}$$

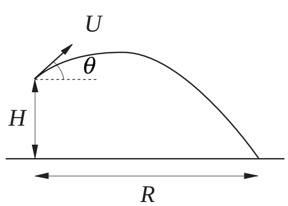


Hence show that the maximum range up the plane is

$$\frac{U^2}{g(1+\sin\alpha)}$$

and is achieved for an angle of projection which bisects the angle between the line of greatest slope of the plane and the vertical through the point of projection.

5. A shot putter can release the shot at a height H above the ground with speed U.



Show that, when the shot is projected at an angle θ to the horizontal, it hits the ground at a horizontal distance R from the point of projection, where R is given by

$$R\tan\theta - g\frac{R^2}{2U^2}\sec^2\theta + H = 0\,.$$

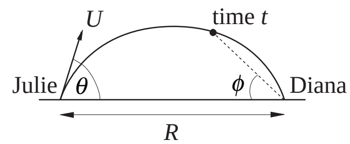
Show that R has a maximum as a function of θ when

$$\tan\theta = \frac{U^2}{gR}\,.$$

Hence show the maximum range is

$$\frac{U}{g}\left(U^2 + 2gh\right)^{\frac{1}{2}}.$$

6. Julie throws a ball with speed U at an angle θ to the horizontal. Diana, who is at a distance R from Julie, catches the ball at the same height above the ground as Julie throws it.



Show that at any time t the tangent of the angle of elevation of the ball relative to Diana, i.e. the angle ϕ with the horizontal made by the line joining the ball's position at time t to the point where Diana catches it, is given by

$$\frac{gt}{2U\cos\theta}$$

7. A stone is thrown with speed 15 ms^{-1} from a cliff 40 m high to land in the sea at a distance of 30 m from the cliff. Show that there are two possible angles of projection and that they make a right angle.

8. Stones are thrown with speed 15 ms^{-1} to clear a wall of height 5 m at a distance of 15 m from the point of projection. Show that no stone can land behind the wall within a distance of 3 m.

9. A fireman is directing water into a window at a height of 10 m above the ground. Because of the heat he wants to stand as far back from the window as possible. The speed with which the water leaves the hose is 50 ms^{-1}. How far back can he stand?

10. (In this question you should assume g is 9.8 ms^{-2})
A particle P is projected from a point O on level ground with speed 50 ms^{-1} at an angle $\sin^{-1}\left(\frac{7}{25}\right)$ above the horizontal. Find
(a) the height of P at the point where its horizontal displacement from 0 is 120 m,
(b) the speed of P two seconds after projection,
(c) the times after projection at which P is moving at an angle of $\tan^{-1}\left(\frac{1}{4}\right)$ to the ground.

(AEB)

11. (In this question you should assume $g = 9.8 \text{ ms}^{-2}$)

At time $t = 0$ a particle is projected from a point O with speed 49 ms^{-1} and in a direction which makes an acute angle θ with the horizontal plane through O. Find, in terms of θ, an expression for R, the horizontal range of the particle from O.

The particle also reaches a height of 9.8 m above the horizontal plane through O at times t_1 seconds and t_2 seconds. Find, in terms of θ, expressions for t_1 and t_2.

Given that $t_2 - t_1 = \sqrt{17}$ seconds, find θ.

Hence show that $R = \dfrac{245\sqrt{3}}{2}$.

(AEB)

12. A particle P is projected at time $t = 0$ in a vertical plane from a point O with speed u at an angle α above the horizontal. Obtain expressions for the horizontal and vertical components of

(a) the velocity of P at time t;

(b) the displacement, at time t, of P from O.

Given that the particle strikes the horizontal plane through O at time T show that

$$T = \frac{2u\sin\alpha}{g}.$$

Find, in terms of g and T, the maximum height that P rises above the horizontal plane through O.

Given also that, at time $\dfrac{3T}{4}$, the particle is moving at right angles to its initial direction, find $\tan\alpha$.

(AEB)

13. A particle projected from a point O on level ground first strikes the ground again at a distance $4a$ from O after time T. Find the horizontal and vertical components of its initial velocity. (AEB)

14. A particle P is projected from a point O on a horizontal plane with speed v in the direction making an angle α above the horizontal. Assuming that the only force acting is that due to gravity, write down expressions for the horizontal and vertical displacements of P from O at a time t after projection. Given that P lands on the horizontal plane at the point A, show that $\text{OA} = 2v^2 \sin\alpha\cos\alpha / g$.

Find the height above the plane of the highest point B of the path of P.

A second particle Q is projected from O with speed v in the direction OB and lands on the plane at C. Find OC.

Find also the value of tan α so that A coincides with C.

(AEB)

15. A particle is projected at time $t = 0$ with speed 49 ms^{-1} at an angle α above the horizontal. The horizontal and vertical displacements from 0, the point of projection, at time t are x m and y m respectively. Obtain x and y in terms of α, g and t and hence deduce that, when $x = 140$ and $g = 9.8 \text{ ms}^{-2}$, $y = 140\tan\alpha - 40(1 + \tan^2\alpha)$.

Find the numerical values of the constants a and b so that this equation can be re-written as

$$y = a - 40(\tan\alpha - b)^2.$$

The particle has to pass over a wall 20 m high at $x = 140$, find

(a) the value of tan α such that the particle has the greatest clearance above the wall,

(b) the two values of tan α for which the particle just clears the wall.

(AEB)

16. Unit vectors **i** and **j** are defined with **i** horizontal and **j** vertically upwards. At time $t = 0$ a particle P is projected from a fixed origin O with velocity $nu(3\mathbf{i} + 5\mathbf{j})$, where n and u are positive constants. At the same instant a particle Q is projected from the point A, where $\overrightarrow{\text{OA}} = a(16\mathbf{i} + 17\mathbf{j})$ with a being a positive constant, with velocity $u(-4\mathbf{i} + 3\mathbf{j})$.

(a) Find the velocity of P at time t in terms of n, u, g and t. Show also that the velocity of P relative to Q is constant and express it in the form $p\mathbf{i} + q\mathbf{j}$.

(b) Find the value of n such that P and Q collide.

(c) Given that P and Q do not collide and that Q is at its maximum height above A when at a point B, find, in terms of u and g, the horizontal and vertical displacements of B from A.

(AEB)