7 NETWORK FLOWS

Objectives

After studying this chapter you should

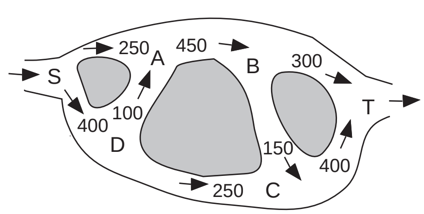
- be able to draw network diagrams corresponding to flow problems;
- be able to interpret networks;
- be able to find optimum flow rates in a network, subject to constraints;
- be able to use the labelling algorithm to find the maximum flow rate in a network;
- be able to interpret the analysis of a network for real life problems.

7.0 Introduction

There are many situations in life which involve flow rates; some are self-evident, such as traffic flow or the flow of oil in a pipeline; others have the same basic structure but are less obviously flow problems - e.g. movement of money between financial institutions and activity networks for building projects. In most of the problems you will meet, the objective is to maximise a flow rate, subject to certain constraints. In order to get a feel for these types of problem, try the following activity.

Activity 1

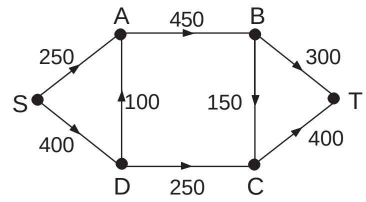
This diagram represents a road network. All vehicles enter at S and leave at T. The numbers represent the maximum flow rate in vehicles per hour in the direction from S to T. What is the maximum number of vehicles which can enter and leave the network every hour?



Which single section of road could be improved to increase the traffic flow in the network?

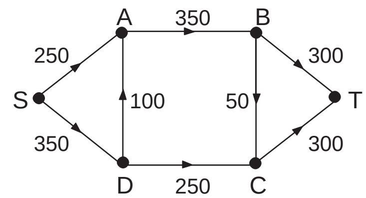
7.1 Di-graphs

The network in the previous activity can be more easily analysed when drawn as a graph, as shown opposite.



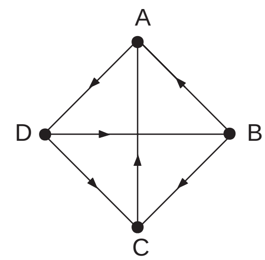
The arrows show the flow direction; consequently this is called a **directed graph** or **di-graph**. In this case the edges of the graph also have **capacities** : the maximum flow rate of vehicles per hour. The vertices S and T are called the **source** and **sink**, respectively.

You should have found that the maximum rate of flow for the network is 600. This is achieved by using each edge with flows as shown.



Notice that some of the edges are up to maximum capacity, namely SA, BT, DA and DC. These edges are said to be **saturated**. Also, at any vertex, other than S or T, in an obvious sense, the **inflow** equals the **outflow**.

Di-graphs for some situations show no capacities on the edges. For example, suppose you have a **tournament** in which four players each play one another. If a player A beats a player B then an arrow points from A to B. In the diagram opposite you can see that A beats D, but loses to B and C.

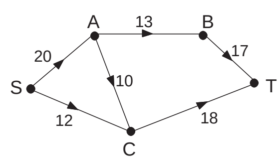


In what follows, the term **network** will be used to denote a directed graph with capacities.

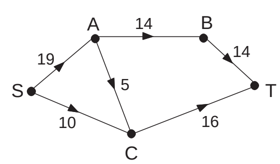
Exercise 7A

1. The diagrams below show maximum flow capacities in network N_1, and actual intended flows in N_2.

N_1



N_2



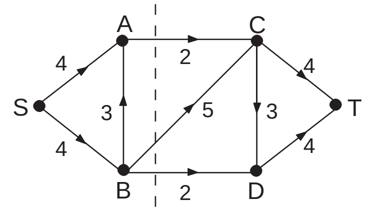
What errors have been made in constructing N_2? Draw a new network which has a maximum flow from S to T.

2. Draw a network representing the results in the tournament described by this table.

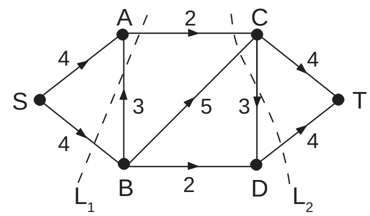
	A	**B**	**C**	**D**	**E**
A	•	X	O	X	O
B	O	•	X	X	X
C	X	O	•	O	X
D	O	O	X	•	X
E	X	O	O	O	•

X denotes a win. O denotes a loss.
For example, the X shows that B beats C.

7.2 Max flow - min cut



The main aim is to find the **value** of the maximum **flow** between the source and sink. You will find the concept of the **capacity** of a **cut** very useful. The network opposite illustrates a straightforward flow problem with maximum allowable flows shown on the edges.



The dotted line shown in the first diagram illustrates one possible **cut**, which separates S from T. Its **capacity** is defined as the sum of the maximum allowable flows across the cut; i.e. $2+5+2=9$. There are many possible cuts across the network. Two more are shown in the second diagram. For L_1, the capacity is

$$2+0+4=6.$$

The reason for the zero is as follows: the flows in AC and SB cross the cut from left to right, whereas the flow in BA crosses from right to left. To achieve maximum flow across the cut the capacity of BA is not used.

Similarly for L_2, the capacity is given by

$$2+5+0+4=11.$$

Activity 2

For the network shown above, find all possible cuts which separate S from T, and evaluate the capacity of each cut. What is the minimum capacity of any cut?

What do you notice about the capacities?

Activity 3

Find the maximum flow for the network shown above. What do you notice about its value?

The activities above give us a clue to the max flow-min cut theorem. You should have noticed that the maximum flow found equals the cut of minimum capacity. In general,

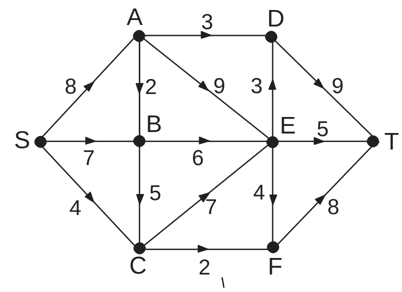
value of any flow $\leq$ capacity of any cut

and equality occurs for maximum flow and minimum cut; this can be stated as

$$\boxed{\text{maximum flow} = \text{minimum cut.}}$$

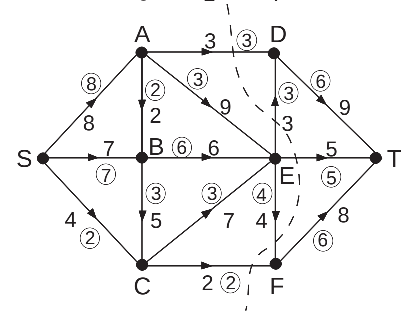
Example

For the network opposite, find the value of the maximum flow and a cut which has capacity of the same value.



Solution

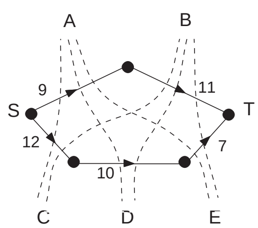
By inspection, the maximum flow has value 17; this is illustrated by the circled numbers on the network opposite. Also shown is a cut of the same capacity.



If you can find a cut and flow of the same value, can you be sure that you have found the maximum flow?

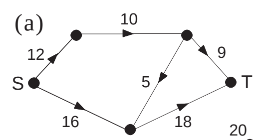
Exercise 7B

1. The network below shows maximum capacities of each edge. Draw up a table showing the values of all the cuts from A, B to C, D, E. Which is the minimum cut? Draw the network with flows which give this maximum total flow.

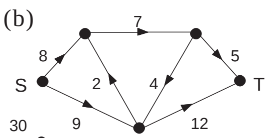


2. Find a minimum cut for each of these networks. The numbers along the edges represent maximum capacities.

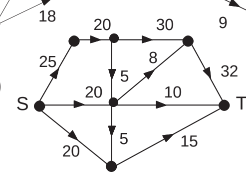
(a)



(b)



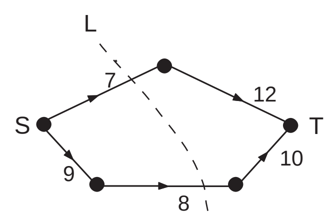
(c)



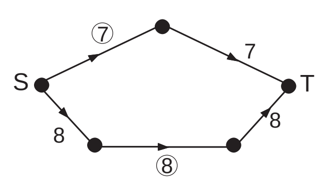
3. For each of the networks in Question 2 try to find values for the flows in each edge which give the maximum overall flow.

7.3 Finding the flow

You may have noticed that the minimum cut is coincident with edges which have a flow equal to their maximum capacity.



The diagram opposite shows a network with its allowable maximum flow along each edge. The minimum cut is marked L. It has a capacity of 15. This line cuts the edges with capacities 7 and 8. The actual maximum flow of value 15 is shown in the diagram, and it should be noted that the minimum cut only passes through edges that are saturated (or have zero flow in the opposing direction).

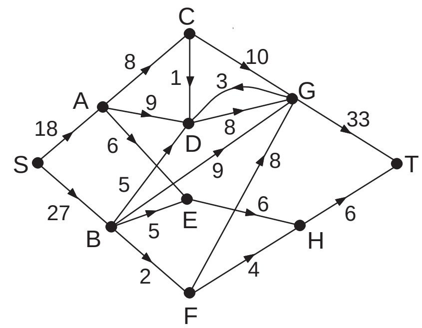


This information should help you to confirm maximum flows. Note that in some cases there is more than one possible pattern for the flows in the edges which give the overall maximum flow.

Activity 4

By trial and error, find the maximum possible flow for the network opposite.

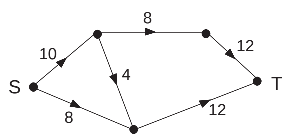
Find a cut which has a capacity equal to the maximum flow (you might find it helpful to mark each edge which is satisfied by the maximum flow - the minimum cut will only cut saturated edges or edges with zero flows in the opposing direction.)



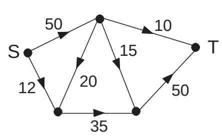
Exercise 7C

1. Find the maximum flow for each of these networks, and show the minimum cut in each case.

(a)



(b)



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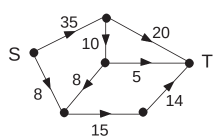


2. A network has edges with maximum capacities as shown in this table.

	S	A	B	C	D	E	T
S	•	40	40	•	•	•	
A	•	•	•	•	15	20	
B	•	•	•	45	•	•	
C	•	•	•	•	•	•	50
D	•	•	10	15	•	•	15
E	•	•	•	•	•	•	25

The letters refer to vertices of the network, where S and T are the source and sink respectively.

Draw a diagram of the network.

Find a maximum flow for the network, labelling each edge with its actual flow.

7.4 Labelling flows

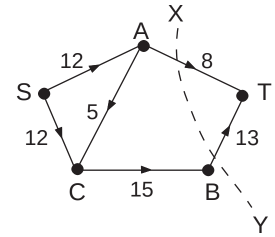
So far you have no method of actually finding the maximum flow in a network, other than by intuition.

The following method describes an algorithm in which the edges are labelled with artificial flows in order to optimise the flow in each arc.

An example follows which shows the use of the **labelling algorithm**.

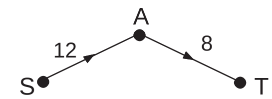
Example

The network opposite has a maximum flow equal to 21, shown by the cut XY. When performing the following algorithm you can stop, either when this maximum flow has been reached or when all paths from S to T become 'saturated'.

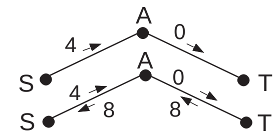


1. Note that there are four possible paths from S to T, namely SAT, SCBT, SACBT and SCAT. (Note that at this stage, the directions of the flows are ignored

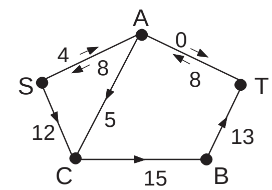
2. Begin with any of these, say SAT, as in the diagram opposite. The maximum flow is restricted by AT, so label each edge with its **excess capacity**, given that AT carries its marked capacity, as shown.



3. Both flows could be reduced by up to 8 (the capacity of edge AT). Show it as a potential backflow in each edge.

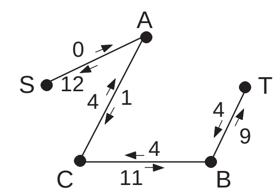


4. Now add this section back on to the original network as shown and choose another route, say SACBT.

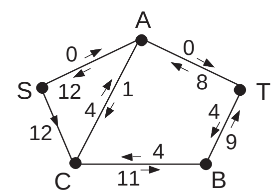


   Of the possible flows, $S \to A$, $A \to C$, $C \to B$ and $B \to T$ note that the lowest is 4 and this represents the maximum flow through this path, as shown.

5. As before, each edge in the path SACBT is labelled with its excess capacity (above 4), and the reverse flows, noting that the sum of the forward and reverse flows always equals the original flow, as shown opposite. Note particularly that the excess flow in SA has now dropped to zero.

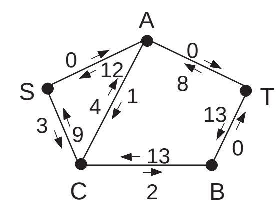


   The resulting network is shown opposite.



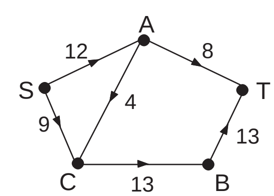
6. Continue by choosing a third path, say SCBT, and inserting artificial forward and backward excess flows.

   The network is shown opposite.



   There is one more route namely SCAT, but it is unnecessary to proceed with the process because the flows to T from A and B are saturated, shown by zero excess flow rates. This means that the flow can increase no further.

7. The excess flows can be subtracted from the original flows to create the actual flows or you can simply note that the back flows give the required result - but with the arrows reversed. The final result is shown opposite.

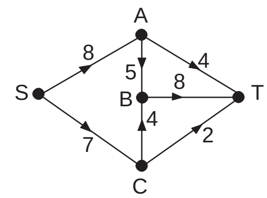


The method looks quite complicated, but after a little practice you should become quite adept at it.

The next example shows how all possible paths sometimes need to be considered.

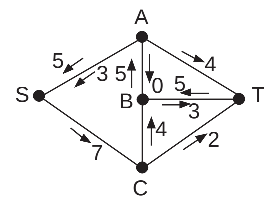
Example

Use the labelling procedure to find the maximum flow from S to T in the network shown opposite.



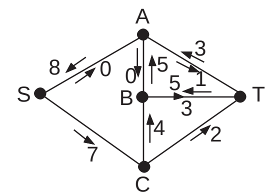
Solution

1. Possible paths SAT, SABT, SABCT, SCT, SCBT, SCBAT.

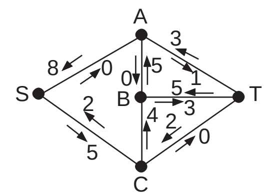


2. Start with SABT - possible to have flow of 5 units, and mark excess capacity in SA and BT and potential backflow.

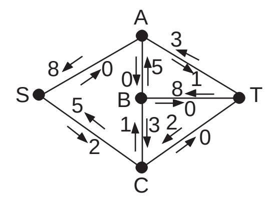
3. Now consider the path SAT - a further 3 units can flow along this path, as shown by the backflows.



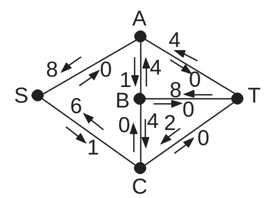
4. Now consider SCT - there is a possible flow of 2 units.



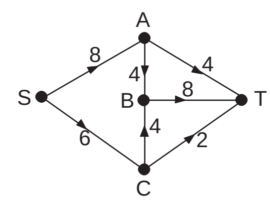
5. A further flow of 3 units is possible along the path SCBT.



6. There is one more path to consider (since there is still excess flow along AT) namely SCBAT - this can take a further one unit (note the way the backflow and excess capacity is shown on AB).



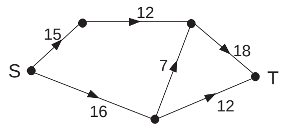
7. No more flow is possible (all areas into T have zero excess capacity), so the maximum flow, as shown opposite, has been achieved.



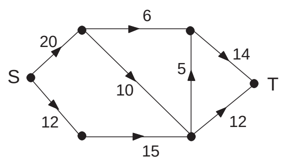
Exercise 7D

1. Use the labelling algorithm in order to find the maximum flow in each of these networks, given the maximum capacity of each edge.

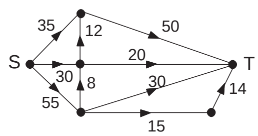
   (a)



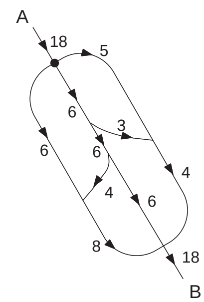
   (b)



   (c)

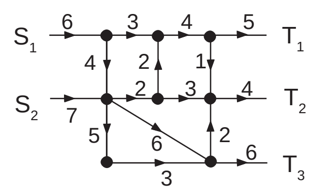


2. There are a number of road routes from town A to town B as shown in the diagram below. The numbers show the maximum flow rate of vehicles in hundreds per hour. Find the maximum flow rate of vehicles from A to B. Suggest a single road section which could be widened to improve its flow rate. How does this affect traffic flow on other sections, if the network operates to its new capacity?



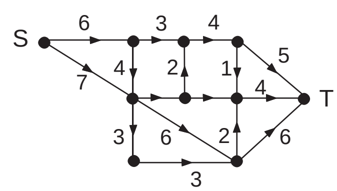
7.5 Super sources and sinks

Many networks have multiple sources and/or sinks. A road network with two sources and three sinks is shown opposite.



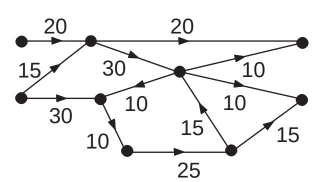
The problem of finding the maximum flow can be quite easily dealt with by creating a single **super source** S and a single **super sink** T.

The resulting network is as shown and the usual methods can now be applied.



Activity 5

Add a super source and a super sink to this network, in which maximum capacities are shown, and then use the labelling algorithm to find a maximum flow through it.

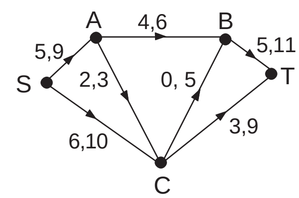


Activity 6

Investigate the traffic flow in a small section of the road network near to you, for which you could estimate maximum flows in each road.

7.6 Minimum capacities

Sometimes edges in networks also have a minimum capacity which has to be met. In the diagram opposite, for example, edge AB has a maximum capacity of 6 and a minimum of 4. The flow in this edge must be between 4 and 6 inclusive.

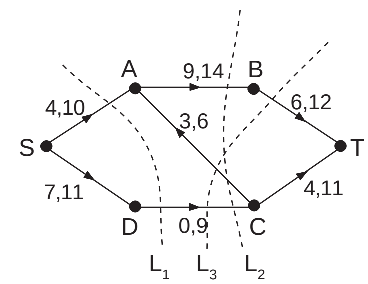


Activity 7

Find the maximum flow in the network shown above.
Investigate how the max flow - min cut theorem can be adapted for this situation.

Example

Find the maximum flow for the network shown opposite.



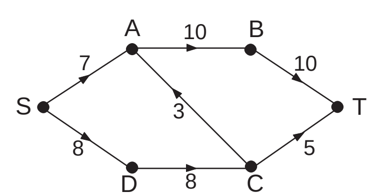
Solution

In order to find a minimum cut, the max flow - min cut theorem is adapted so that you **add** upper capacities of edges along the cut directed from S to T, but **subtract** lower capacities of edges directed from T to S.

For this network, cut L_1 has a value $10+9=19$, but cut L_2 has a value of $14+9-3=20$, since edge CA crosses L_2 from T to S. In fact, L_3 is the minimum cut - with a value of $12-3+9=18$, so you are looking for a maximum flow of 18.

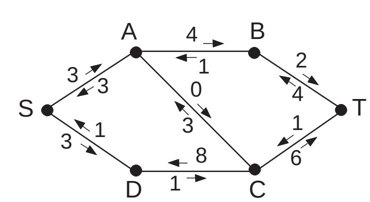
If you are familiar with the labelling algorithm, here is a slightly quicker version .

1. Begin with **any** flow. The one shown opposite will do. Note that none of the upper or lower capacities of the edges has been violated. It is not the best because the flow is only 15.



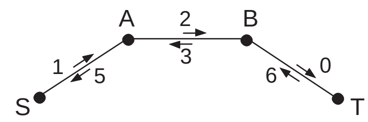
2. For each edge, insert the potential excess flow and the corresponding back flow. For example, BT carries a flow of 10 at present. It could be 2 more and it could be 4 less, since the minimum and maximum flows in the edge are 6 and 12.

   This has been done for all the edges in the network resulting in the diagram opposite.

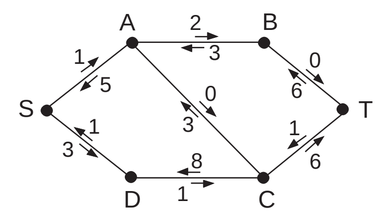


3. Now look for a path in which the flow (from S to T) can be improved. Consider SABT. The lowest excess capacity in these three edges is 2 (in BT) so the flow in each edge can be improved by this amount.

   SABT is called a **flow augmenting path** because its overall excess flow can be reduced. The reverse flow has to be increased by 2 to compensate. The path now looks like this.



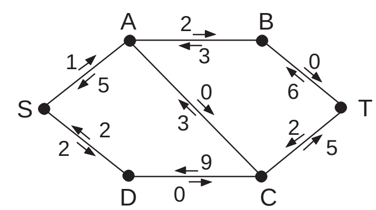
4. This can now be added to the network as shown opposite.



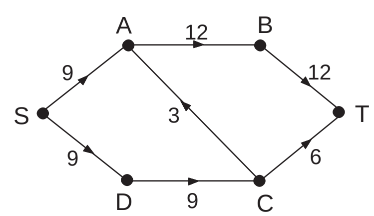
5. Now look for another path which can be augmented (improved).

   SACT cannot, since AC has a flow of zero from S to T.

   SDCABT cannot, since BT has a zero flow. The only possibility is SDCT. This can be improved by an increase of one. The result is shown here.



6. Now subtract the excess ($S \rightarrow T$ directed) flows from their maximum values. So, for example, AB becomes $14 - 2 = 12$. The final network - as shown opposite - has a flow of 18 as required.

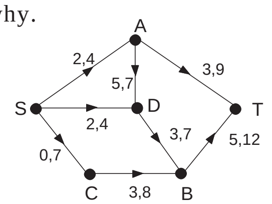


Exercise 7E

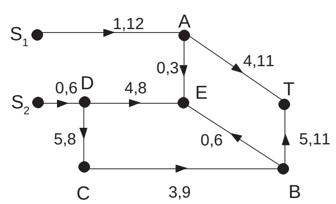
1. For the network of Activity 7 find any flow in the network and then use the labelling algorithm to find the maximum flow.

2. Which, if any, of the following networks showing upper and lower capacities of edges has a possible flow? If there is no possible flow, explain why.

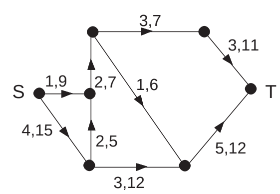
   (a)



   (b)



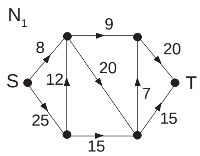
3. Find a maximum flow from S to T in this network showing upper and lower capacities.

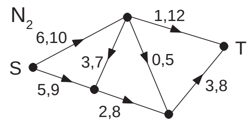


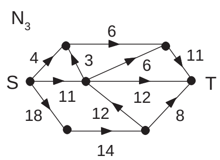
7.7 Miscellaneous Exercises

1. Find a minimum cut for each of the following networks.

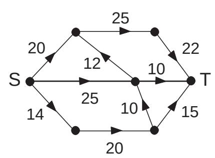
   In N_1 and N_3 edges can carry the maximum capacities shown. In N_2 the minimum and maximum capacities of the edge are shown.



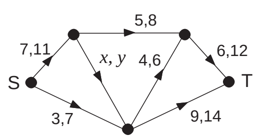




2. Use the labelling algorithm to find a maximum flow in this network, which shows maximum capacities.



3. Which of these values of x and y gives a possible flow for the networks shown below with upper and lower capacities?

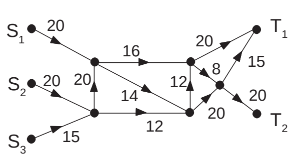


   (a) 7, 10 (b) 3, 12

   (c) 4, 5 (d) 1, 8

   For these cases find a maximum flow for the networks.

4. By creating a super source and a super sink find a maximum flow for this network which shows maximum capacities.

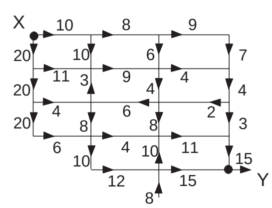


5. The table shows the daily maximum capacity of coaches between various cities (in hundreds of people).

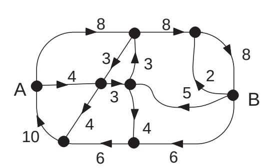
   Draw a network to show the capacities of the routes from London through to Newcastle.A festival is taking place in Newcastle. Find the maximum number of people who can travel by coach from London for the festival. Investigate what happens when there is a strike at one of the coach stations, say Liverpool.

From \ To	Lon	Bir	Man	Lds	Lpl	New
London	•	40	•	20	•	•
Birmingham	•	•	10	15	12	•
Manchester	•	•	•	12	•	15
Leeds	•	•	•	•	•	30
Liverpool	•	•	7	•	•	8

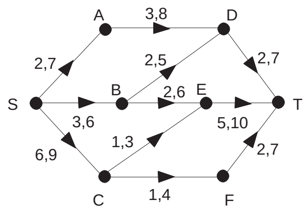
6. Find a maximum traffic flow on this grid-type road system from X to Y, in which maximum flow rates are given in hundreds of vehicles per hour.



7. The following underground map shows a 'circular route' with 8 stations. Trains travel only in the direction shown. The capacities indicate the maximum number of trains per hour which can pass along each section. **At least** one train per hour must travel along each section of track. A train can carry 500 passengers. Find the maximum number of passengers which can flow from A to B.
   Note that A and B are not sources or sinks. The number of trains in the system must always remain constant.



8. Consider the following network with lower and upper capacities.

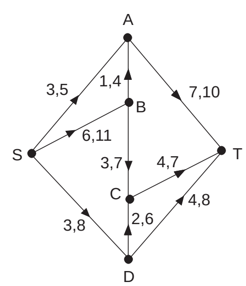


(a) Find by inspection a feasible flow from S to T whose value does not exceed 14.

(b) By looking for flow-augmenting paths (or otherwise), find a maximum flow in this network.

(c) Find a corresponding minimum cut, and check that its capacity is the same as the value of the maximum flow found in part (b).

9. Consider the following network with lower and upper capacities:

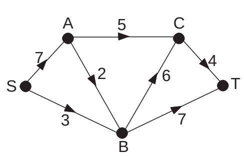


(a) Find by inspection a feasible flow from S to T whose value does not exceed 17.

(b) By looking for flow-augmenting paths (or otherwise), find a maximum flow in this network.

(c) Find a corresponding minimum cut, and check that its capacity is the same as the value of the maximum flow found in part (b).

10. Consider the following network, in which each arc is labelled with its capacity:

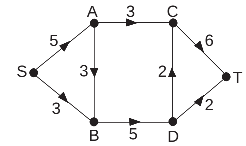


(a) Find a flow of value 9 from S to T, and draw a diagram showing the flow in each arc.

(b) Find a cut of capacity 9.

(c) What is the value of a maximum flow? (Give a brief reason for your answer.)

11. Consider the following network, where each arc is labelled with its capacity.



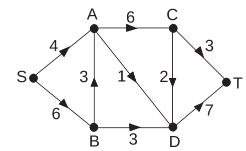
(a) Find a flow of value 7, and draw a diagram showing the flow in each arc.

(b) Find a cut of capacity 7.

(c) What is the value of a maximum flow?

(In part (c), you should give a brief reason for your answer.)

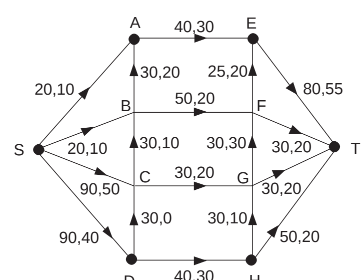
12. Verify that the max-flow min-cut theorem holds for the following network:



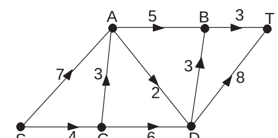
13. The network below shows the maximum and minimum flow allowed along each arc of the network.

(a) Ignoring the minimum flow constraints, find a feasible flow between S and T of value 140.

(b) Find the maximum flow, when both maximum and minimum constraints operate. Explain why your flow is a maximum flow.



14. In the following basic network, each arc is labelled with its capacity:

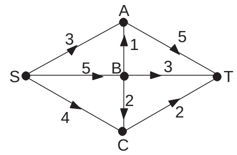


(a) Write down, or indicate on a diagram, a cut of capacity 9, separating S from T.

(b) Find a flow of value 9 from S to T, and draw a diagram showing the flow in each arc.

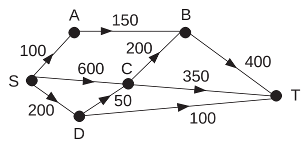
(c) What is the value of a maximum flow? (Give a brief reason for your answer.)

15. (a) In the following basic network, find a flow of value k from S to T, and cut with capacity k, for the same value of k.



(b) Is the flow in part (a) a maximum flow? (Give a reason for your answer.)

16. The network below shows the maximum rates of flow (in vehicles per hour) between towns S, A, B, C, D and T in the direction from S to T.



(a) By choosing a minimum cut, or otherwise, find the maximum traffic flow from S to T. Give the actual rates of flow in each of the edges BT, CT and DT when this maximum flow occurs.

(b) When a maximum flow occurs from S to T, how many of those vehicles per hour pass through C?

(c) It is decided to reduce the traffic flow through C (in the direction from S to T) to a maximum of 480 vehicles per hour. In order to maintain the same maximum flow from S to T the capacity of a single edge is to be increased. Which edge should be chosen, and by how much must its capacity be increased?

(AEB)