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# 6 PLANAR GRAPHS

## Objectives

After studying this chapter you should

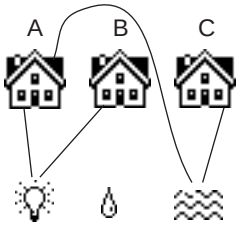
- be able to use tests to decide whether a graph is planar;
- be able to use an algorithm to produce a plane drawing of a planar graph;
- know whether some special graphs are planar;
- be able to apply the above techniques and knowledge to problems in context.

## 6.0 Introduction

This topic is introduced through an activity.

### Activity 1

A famous problem is that of connecting each of three houses, as shown opposite, to all three services (electricity, gas and water) with no pipe/cable crossing another.




Try this problem. Four of the nine lines needed have already been put into the picture.

Investigate the problem for different numbers of houses and services.

What happens if the scene is on the surface of a sphere (which in reality it is) or on a torus (a ring doughnut!) or on a Möbius strip?

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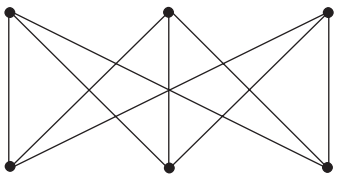

You will have found graphs which can be completed without their edges crossing and some graphs which cannot. If a graph can be drawn in the plane (on a sheet of paper) without any of its edges crossing, it is said to be **planar**.



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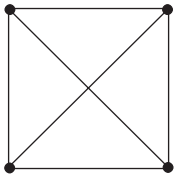
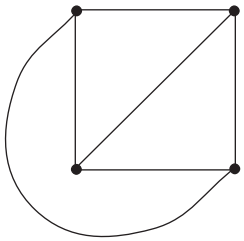
The graph shown on the right is planar, although you might not think so from the first diagram of it. The next diagram is the same graph and confirms that it is planar. This diagram is called a **plane drawing** of the graph.



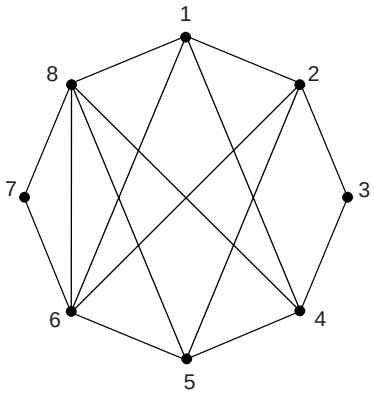


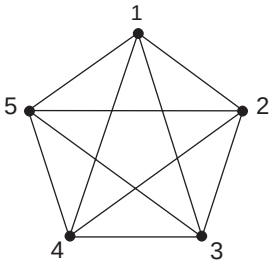
As you should have found in the activity above, the graph shown on the previous page, which represents the services problem, cannot in fact be drawn without crossing edges and is therefore described as **non-planar**. This has repercussions for the electronics industry, because it means that a simple circuit with six junctions and wires connecting each of three junctions to each of the other three junctions cannot be made without cross-overs. In an integrated circuit within a 'chip', this would mean two 'layers' of wires.

# 6.1 Plane drawings

## Activity 2

Try re-drawing the two graphs shown on the right so that no edges cross.




The second of the two graphs is called $K_5$, the complete graph with five vertices: each vertex is joined to every other one by an edge. Of course, $K_6$, $K_7$, ..., are similarly defined. Although $K_5$ looks simpler than the one shown above it, it is in fact non-planar, whereas the one above it is planar.

Later on it will be proved that both $K_5$ and another graph called $K_{3,3}$ (which is the one associated with the gas, water and electricity problem in Activity 1) are non-planar, and you will see the significance of this when you look at Kuratowski's Theorem later in this chapter.

## *Exercise 6A*

1. Sketch the graphs of $K_4$ and $K_6$. Are they planar? For which values of $n$ do you think $K_n$ is planar?
2. For which values of positive integers is $K_n$ Eulerian?



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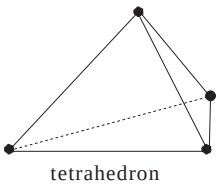
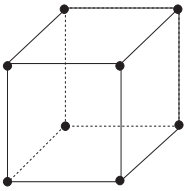
3. There are five so-called Platonic solids with a very regular structure. Graphs based on the first three are shown below.

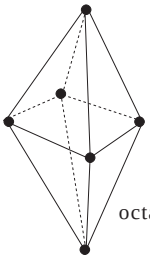
   Make plane drawings of each, if possible.



tetrahedron



cube



octahedron

4. The graph $K_n$ can be used to represent the games played during a 'round robin' tournament in which each player plays every other player.

   How many games take place when there are

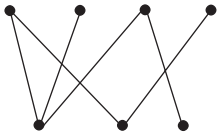
   (a) 5 players

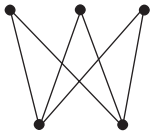
   (b) $n$ players?

5. Draw a graph with six vertices, labelled 1 to 6, in which two vertices are joined by an edge if, and only if, they are **co-prime** (i.e. if they have no common factor greater than 1). Is the graph planar?

# 6.2 Bipartite graphs

The graph associated with the activity in Section 6.1 is called a **bipartite** graph. Such graphs consist of two sets of vertices, with edges only joining vertices between sets and not within a set. The diagrams opposite are of bipartite graphs. In the second one the two sets of vertices contain three vertices and two vertices and every vertex in the first set is joined to every vertex in the second: this graph is called $K_{3,2}$ (and, of course, $K_{r,s}$ could be defined similarly for any positive integers $r$ and $s$).





## *Exercise 6B*

1. Sketch $K_{3,4}$ and $K_{4,2}$.
2. How many edges are there in general in the graph $K_{r,s}$?
3. Two opposing teams of chess players meet for some games. Show how a bipartite graph can be used to represent the games actually played. If the graph turned out to be $K_{r,s}$ what would it mean?
4. For which $r$ and $s$ is $K_{r,s}$ Eulerian? For which $r$ and $s$ is it semi-Eulerian?



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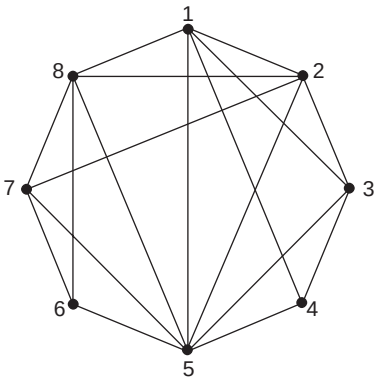
# 6.3 A planarity algorithm

Naturally, for very complicated graphs it would be convenient to have a technique available which will tell you both whether a graph is planar and how to make a plane drawing of it.




## Activity 3

Using the first graph shown opposite as an example, try to develop an algorithm in order to construct a planar graph.

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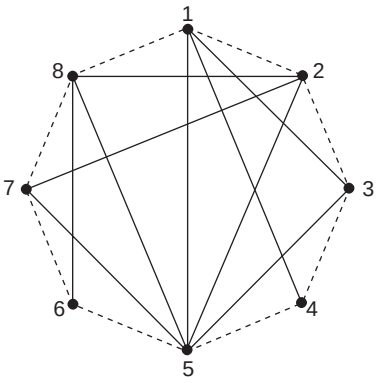
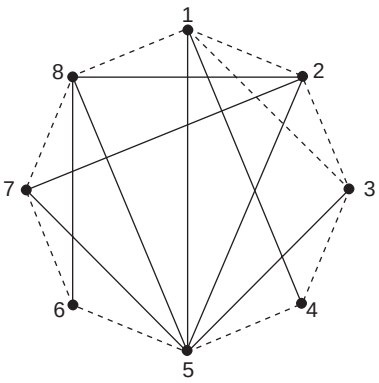
The algorithm described below can be applied only to graphs which have a Hamiltonian cycle; that is, where there is a cycle which includes every vertex of the graph.




The method will be illustrated by applying it to the graph shown in Activity 3, above.

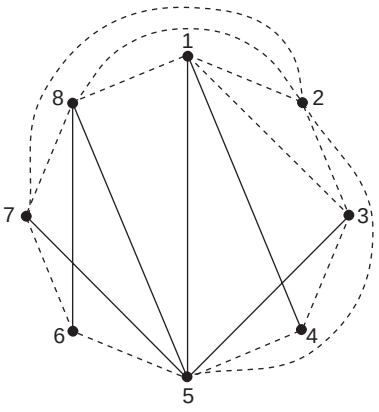
The first stage is to redraw the graph so that the Hamiltonian cycle forms a regular polygon and all edges are drawn as straight lines inside the polygon. The graph used here is already in this form, but for other graphs this stage might involve 'moving' vertices as well as edges.

The edges of the regular polygon now become part of the solution (shown dotted in the second graph ).




The next stage is to choose any edge, say 1 - 3, and decide whether this is to go inside or outside: let's choose inside, as illustrated in the third graph opposite.

Since 1 - 3 crosses 2 - 8, 2 - 7 and 2 - 5, all these edges must go outside as shown.



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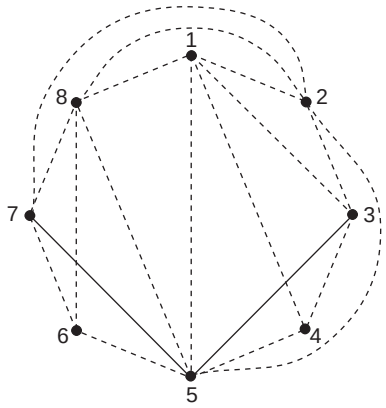
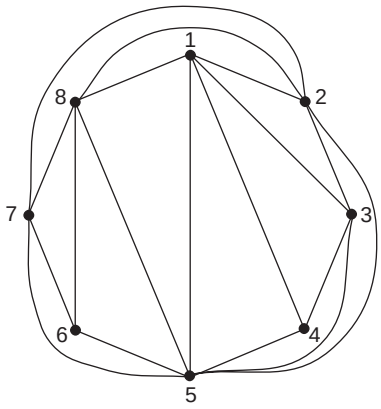
Originally edge 2 - 7 crossed 1 - 4, 1 - 5, 8 - 5 and 8 - 6 , so all these edges must now remain inside (or they would cross 2 - 7 outside).

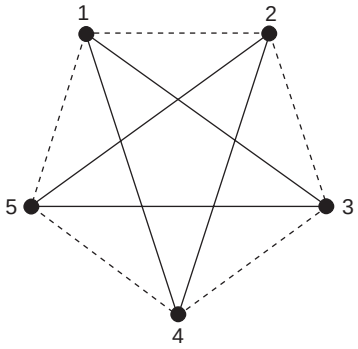
Finally, because 1 - 4 stays inside, 3 - 5 must go outside, and since 8 - 6 stays inside, 7 - 5 must also go outside, as shown.



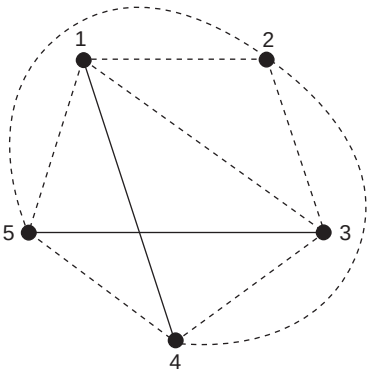

This is now a planar graph, as shown opposite, where the dotted lines have been redrawn as solid lines.




The method illustrated above can also be used to show whether or not a graph is planar. For example, consider $K_5$, and, as before, the regular polygon is first included as part of the solution.




Choose an edge, say 1 - 3, which stays inside. Since this crosses 2 - 5 and 2 - 4, both of these will have to go outside as shown.



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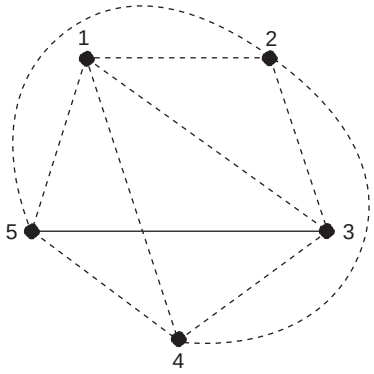
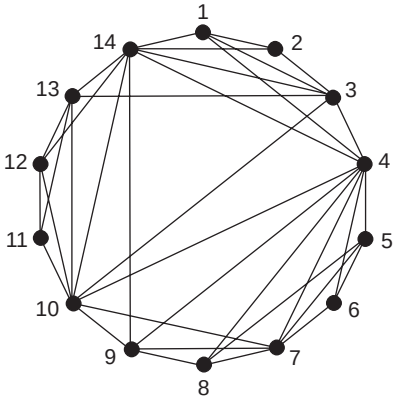



Now 2 - 5 crosses 1 - 4, so 1 - 4 must stay inside, as shown.

Finally, consider edge 3 - 5. Since it crosses 1 - 4, it must go outside; but it also crosses 2 - 4 which is already outside; so 3 - 5 must also go outside! This is a contradiction and it is concluded that the graph is non-planar.

## Example

Use the planarity algorithm to find a plane drawing of the graph opposite.

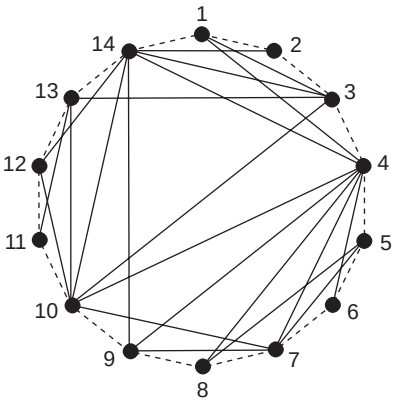



### Solution

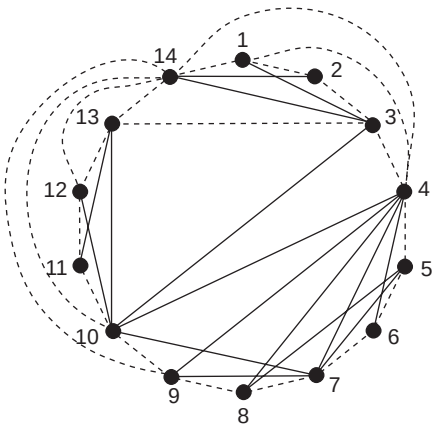
The graph has a Hamiltonian cycle

1 - 2 - 3 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11 - 12 - 13 - 14 - 1

which is part of the solution, as indicated by the dotted lines in the second graph.




Choose any edge, say 13 - 3, and keep it in place (shown dotted). All edges that cross this line must now be put outside.



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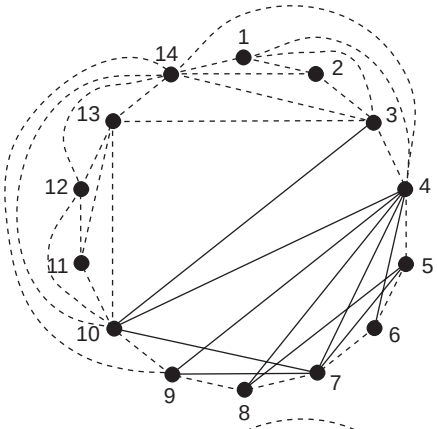
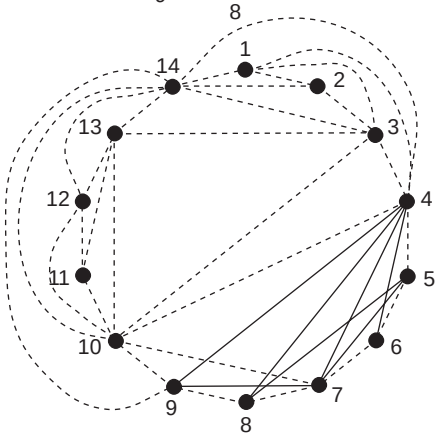
Since 1 - 4 crosses 14 - 3 and 14 - 2, they must go inside, and similarly 1 - 3 must go outside.

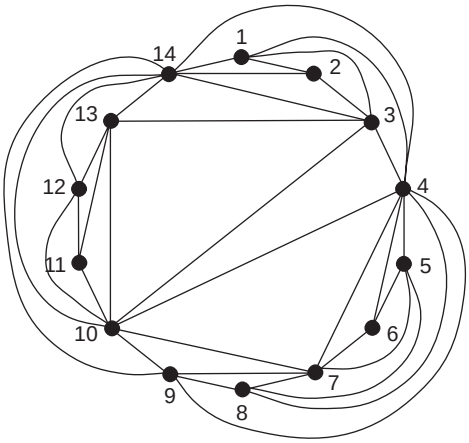



Also,14 - 4 crosses 10 - 3 so 10 - 3 must stay inside; and since 14 - 9 is now outside, 10 - 4 and 10 - 7 must stay inside. Also 14 - 12 outside implies 13 - 10 and 13 - 11 inside, which then means that 12 - 10 must be drawn outside (as shown opposite).




Continuing in this way, 10 - 7 inside means that 9 - 4, 8 - 4, 8 - 5 all go outside; which in turn means that 9 - 7 , 7 - 4 and 6 - 4 go inside. 7 - 5 must go outside.

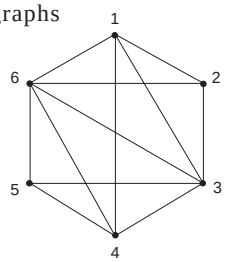
You now have a plane drawing of the graph as shown opposite; the lines are not dotted since this is the final solution.

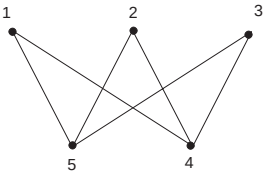



**Is this solution unique?**

## Exercise 6C

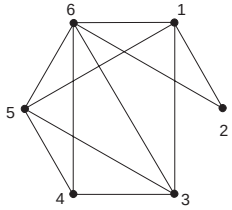
1. Make plane drawings of the following two graphs




2. Show that the graphs of $K_6$ and $K_{3,3}$ are not planar by using the algorithm. (Note that $K_{3,3}$ must first be redrawn to form a regular polygon.)

3. By first redrawing with a regular polygon, use the planarity algorithm to produce a plane drawing of the graph shown below.



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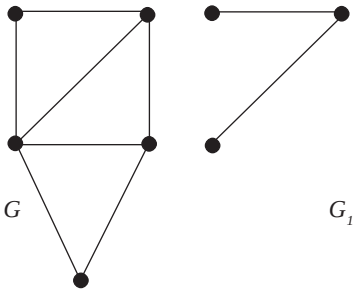
# 6.4 Kuratowski's Theorem

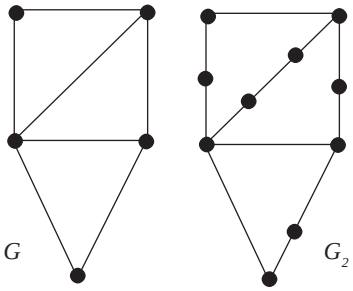
The non-planar graphs $K_5$ and $K_{3,3}$ seem to occur quite often. In fact, all non-planar graphs are related to one or other of these two graphs.




To see this you first need to recall the idea of a **subgraph**, first introduced in Chapter 1 and define a **subdivision** of a graph.

A subgraph is simply a part of a graph, which itself is a graph. $G_1$ is a subgraph of $G$ as shown opposite.

A subdivision of a graph is the original graph with added vertices of degree 2 along the original edges. As shown opposite, $G_2$ is a subdivision of $G$.




When a planar graph is subdivided it remains planar; similarly if it is non-planar, it remains non-planar.

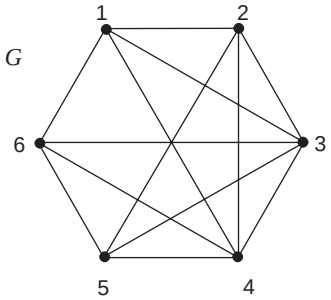
Kuratowski's Theorem states that a graph is planar if, and only if, it does not contain $K_5$ and $K_{3,3}$, or a subdivision of $K_5$ or $K_{3,3}$ as a subgraph.

This famous result was first proved by the the Polish mathematician *Kuratowski* in 1930. The proof is beyond the scope of this text, but it is a very important result.

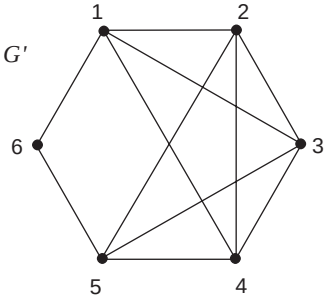
The theorem will often be used to show a graph is non-planar by finding a subgraph of it which is either $K_5$ or $K_{3,3}$ or a subdivision of one of these graphs.

## Example

Graph $G$ has been redrawn, omitting edges 3 - 6 and 4 - 6. Thus $G'$ is a subgraph of $G$.




Also $G'$ without vertex 6 is isomorphic to $K_5$. The addition of vertex 6 makes $G'$ a subdivision of $K_5$. So $G'$, a subdivision of $K_5$, is a subgraph of $G$, and therefore $G$ is non-planar.



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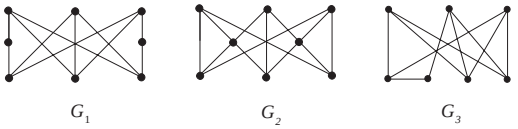
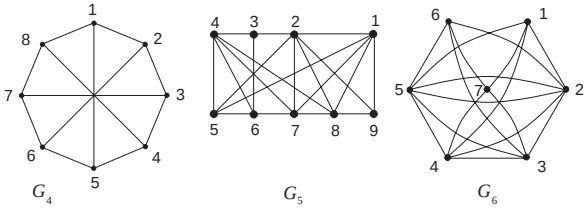
## Exercise 6D

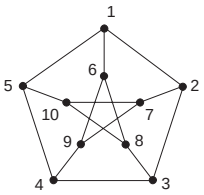
1. Which of these graphs are subdivisions of $K_{3,3}$ and why?




2. Use Kuratowski's Theorem to show that the following graphs are non-planar.

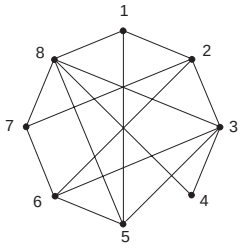



3. Show that this graph, called a Petersen graph, is non-planar.

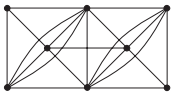



# 6.5 Miscellaneous Exercises

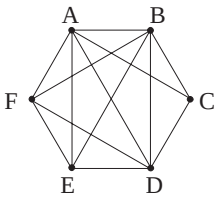
1. Give examples of
   (a) a planar graph in which each vertex has degree 4, and
   (b) a planar graph with six vertices and a shortest cycle of length 4.
2. For which values of $r$, $s$ is the complete bipartite graph $K_{r,s}$ non-planar?
3. The crossing number of a graph is the least number of points at which edges cross. What are the crossing numbers of
   (a) $K_{3,3}$
   (b) $K_6$
   (c) $K_{1,2}$?
4. Use Kuratowski's Theorem in order to prove that the graph $G$ is non-planar.




5. Show that this graph is planar.



6. The graph below represents connections in an electrical circuit. Use a planarity algorithm to decide whether or not it is possible to redraw the connections so that the graph is planar.

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