2 TRAVEL PROBLEMS

Objectives

After studying this chapter you should

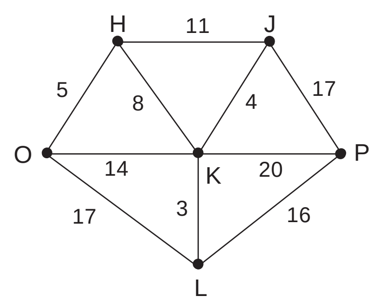
- be able to explain the shortest path, minimum connector, travelling salesman and Chinese postman problems and distinguish between them;
- understand the importance of algorithms in solving problems;
- be able to apply a given algorithm.

2.0 Introduction

In this chapter the ideas developed in Chapter 1 are applied to four important classes of problem. The work will make very little sense unless you have studied Chapter 1 already, but no other mathematical knowledge is required beyond some basic arithmetic. A simple calculator may be helpful for some of the work.

2.1 The shortest path problem

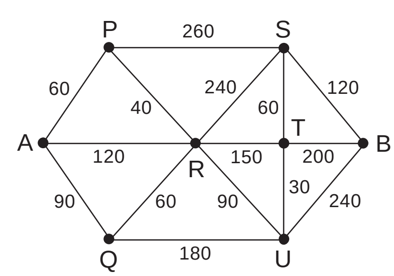
The graph opposite represents a group of towns, with the figures being the distances in miles between them. It may seem strange that the direct edge OK is longer than the two edges OH and HK added together, but in real life this is not unusual. You can probably think of examples in your own area where the signposted route from one place to another is longer than an 'indirect' route via a third place, known only to local people.



Activity 1 Shortest path

Find by trial and error the shortest route from O to P.

The diagram opposite shows a number of villages in a mountainous area and the time (in minutes) that it takes to walk between them. Once again you may notice that there are cases where two edges of a triangle together are 'shorter' than the third - this is perhaps because the third edge goes directly over the top of a mountain, while the first two go around the side.



Activity 2 Shortest path

Find the quickest route from A to B.

You may think that there should be some logical and systematic way of finding a shortest route, without relying on a lucky guess. As the number of vertices increases it becomes more and more important to find such a method; and if the problem is to be turned over to a computer, as is usual when problems such as these arise in real life, it is not just important but essential.

What is actually needed is an **algorithm -** that is, a set of step-by-step instructions that can be applied automatically, without any need for personal judgement. A recipe in a cookery book is a good example of an algorithm, as is a well-written set of directions for finding someone's house. The 'Noughts and Crosses' strategy in Section 1.6 is another example, and any computer program depends on an algorithm of some kind.

Activity 3 Finding an algorithm

On your own or in discussion with another student, try to write an algorithm for solving a shortest path problem. That is, write a set of instructions so that someone who knows nothing about mathematics (except how to do simple arithmetic) can solve any such problem just by following your instructions. Test your algorithm on the examples on the previous page.

The shortest path algorithm

The usual algorithm sometimes known as Dijkstra's method is set out below; you may have come up with something similar yourself, or you may have found a different approach. Study this algorithm anyway, and then decide how you want to handle problems such as these in the future.

Consider two vertex sets: the set S, which contains only the start vertex to begin with, and the set T, which initially contains all the other vertices. As the algorithm proceeds, each vertex in turn will be labelled with a distance and transferred from T to S.

The algorithm runs as follows:

1. Label the start vertex with distance 0.
2. Consider all the edges joining a vertex in S to a vertex in T, and calculate for each one the sum of its length and the label on its S-vertex.
3. Choose the edge with the smallest sum. (If there are two or more with equally small sums, choose any of them at random.)
4. Label the T-vertex on that edge with the sum, and transfer it from T to S.
5. Repeat Steps 2 to 4 until the finish vertex has been transferred to S.
6. Find the shortest path by working backwards from the finish and choosing only those edges whose length is exactly equal to the difference of their vertex labels.

See how the algorithm works on the first example at the beginning of the chapter .

Initially, vertex O is in S, with label 0.

Edges joining {O} to {H, J, K, L, P} are OH $(0+5=5)$, OK $(0+14=14)$ and OL $(0+17=17)$. Choose OH, label H as 5, and transfer H to set S.

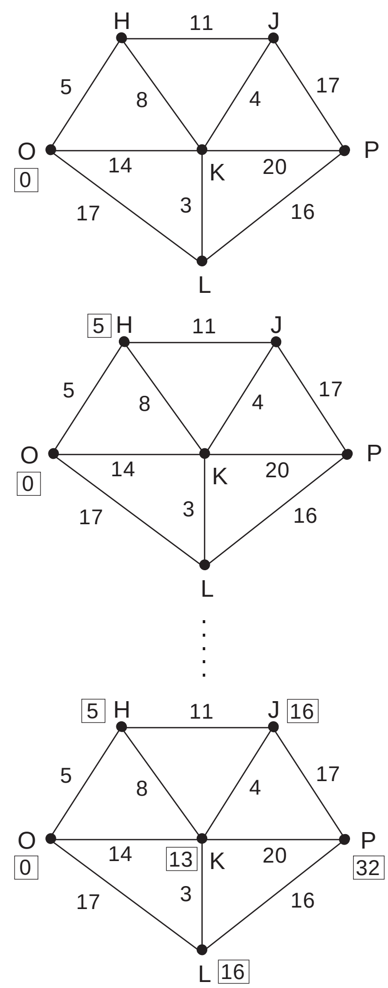
Edges joining {O, H} to {J, K, L, P} are OK (14), OL (17), HJ $(5+11=16)$ and HK $(5+8=13)$. Choose HK, label K as 13, and transfer K to set S.

Edges joining {O, H, K} to {J, L, P} are OL (17), HJ (16), KJ $(13+4=17)$, KP $(13+20=33)$ and KL $(13+3=16)$. There are two equal smallest sums, so choose (say) HJ, label J as 16, and transfer J to set S.

Edges joining {O, H, K, J} to {L, P} are OL (17), KL (16), KP (33) and JP $(16+17=33)$. Choose KL, label L as 16, and transfer L to set S.

Edges joining {O, H, K, J, L} to {P} are KP (33), JP (33) and LP $(16+16=32)$. Choose LP, label P as 32, and transfer P to set S.

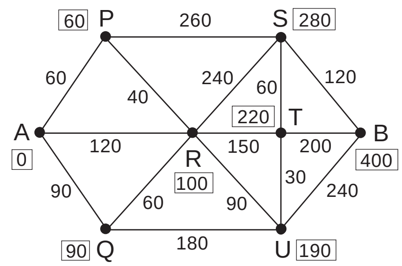
Working back from P, the edges whose lengths are the difference between their vertex labels are PL, LK, KH and HO (HJ would be a forward, not a backward, move), so the shortest path is O - H - K - L - P, which is 32 miles long.



Some stages in the application of the algorithm

The second example can be dealt with similarly. In brief:

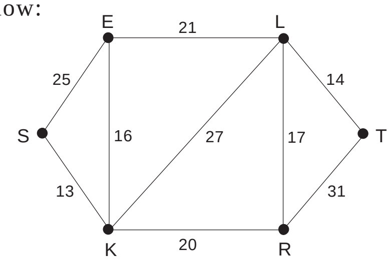
1. Label A as 0.
2. Choose AP and label P as 60.
3. Choose AQ and label Q as 90.
4. Choose PR and label R as 100.
5. Choose RU and label U as 190.
6. Choose UT and label T as 220.
7. Choose TS and label S as 280.
8. Choose SB and label B as 400.

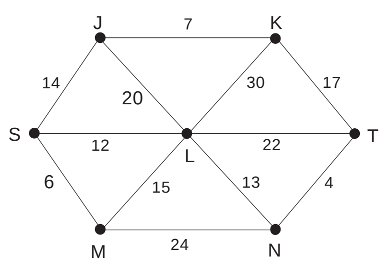


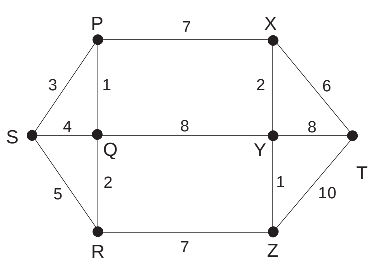
Scan back, choosing BS, ST, TU, UR, RP and PA to obtain the quickest route A - P - R - U - T - S - B, which takes 400 minutes.

Exercise 2A

1. Use the shortest path algorithm to find the shortest path from S to T in each of the diagrams below:



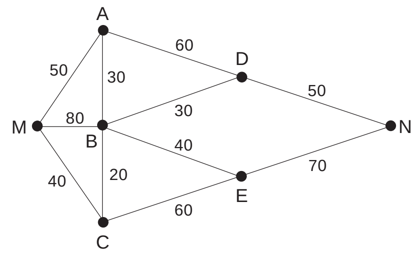




2. The table shows the cost in £ of direct journeys which are possible on public transport between towns A, P, Q, R, S and B. Find the cheapest route by which a traveller can get from A to B.

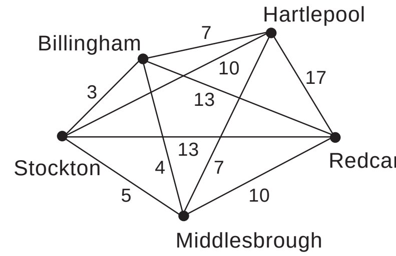
	A	P	Q	R	S	B
A	-	6	4	-	-	-
P	6	-	-	5	6	-
Q	4	-	-	3	7	-
R	-	5	3	-	-	8
S	-	6	7	-	-	5
B	-	-	-	8	5	-

3. The diagram represents the length in minutes of the rail journeys between various stations. Allowing 10 minutes for each change of trains, find the quickest route from M to N.



2.2 The minimum connector problem

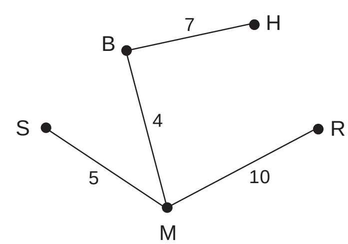
A cable television company wants to provide a service to each of five towns, and for this purpose the towns must be linked (directly or indirectly) by cable. For reasons of economy, the company are anxious to find the layout that will minimise the length of cable needed.



Five towns in Cleveland

The diagram shows in the form of a graph - not a map drawn to scale - the distances in miles between the towns. The layout of cable needs to form a connected graph joining up all the original vertices but, for economy, will not have any cycles. Hence we are looking for a subgraph of the one illustrated which is a tree and which uses all the original vertices: such a subgraph is called a **spanning tree**.

In addition, to find the minimum length of cable needed, from all the possible spanning trees we are looking for one with the total length of its edges as small as possible: this is sometimes known as a **minimum connector**.



The lower diagram shows one of the spanning trees of the graph. It is certainly a tree, and it joins all the vertices using some of the edges of the original graph. Its total length is 26 miles, however, and is not the shortest possible.

Activity 4 Minimum-length spanning tree

Try to find another spanning tree whose total length is minimal.

An oil company has eight oil rigs producing oil from beneath the North Sea, and has to bring the oil through pipes to a terminal on shore. Oil can be piped from one rig to another, and rather than build a separate pipe from each rig to the terminal the company plans to build the pipes in such a way as to minimise their total length.

Activity 5

If the distances in km between the rigs A - H and the terminal T are as shown in the table on the next page, find this minimum total length and say how the connections should be made in order to minimise the total length.

	T	A	B	C	D	E	F	G	H
T	-	120	150	140	120	100	160	70	180
A	120	-	60	60	90	190	210	160	40
B	150	60	-	20	80	180	170	160	50
C	140	60	20	-	40	160	150	140	60
D	120	90	80	40	-	130	70	110	120
E	100	190	180	160	130	-	140	30	220
F	160	210	170	150	70	140	-	150	200
G	70	160	160	140	110	30	150	-	200
H	180	40	50	60	120	220	200	200	-

Like most examples in mathematics books, these problems are oversimplified in various ways. The presentation of the problems implies that junctions can occur only at the vertices already defined, and while this may just possibly be true in the second case (because of the difficulty and expense of maintaining a junction placed on the sea bed away from any rig) it seems unlikely in the first. On the other hand, the terms of the second problem ignore the fact that the costs of laying any pipe depend to some extent on the volume of oil that it is required to carry, and not solely on its length. Even so, these examples serve to illustrate the general principles involved in the solution of real problems.

The first problem above, with just five vertices, can be solved by a combination of lucky guesswork and common sense. The second problem is considerably harder - with nine vertices it is still just about possible to find the minimum connector by careful trial and error, but it is certainly not straightforward. As the number of vertices increases further, so the need for an algorithm grows.

Try to write down your own algorithm for solving problems of this type.

2.3 Kruskal's algorithm

There are two generally-known algorithms for solving the minimum connector problem; your algorithm may be essentially the same as one of these, or it may take a different approach altogether. Whatever the case, work carefully through the next two sections to see how Kruskal's and Prim's algorithms work, and compare the results they give with any that you may have obtained by other methods.

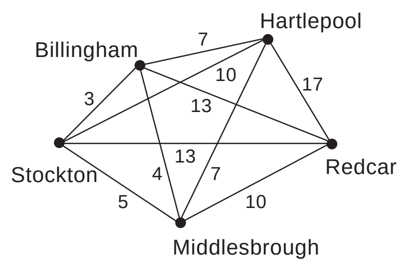
You will recall that the problem is to find a minimum-length spanning tree, and that a spanning tree is a subgraph including all the vertices but (because it is a tree) containing no cycles. **Kruskal's algorithm**, sometimes known as the 'greedy algorithm', makes use of these facts in a fairly obvious way.

The algorithm for *n* vertices is as follows:

1. Begin by choosing the shortest edge.
2. Choose the shortest edge remaining that does not complete a cycle with any of those already chosen. (If there are two or more possibilities, choose any one of them at random.)
3. Repeat Step 2 until you have chosen $n-1$ edges altogether; the result is a minimum-length spanning tree.

Example

Look again at the first problem posed in Section 2.2, the diagram for which is repeated here for convenience.



The shortest edge is BS = 3, so choose BS.

The shortest edge remaining is BM = 4, so choose BM.

The shortest edge remaining is SM = 5, but this completes a cycle and so is not allowed. Next shortest is BH or MH, both = 7; so choose randomly (say) BH.

The shortest edge remaining (apart from SM, which is already excluded) is HM, but this would complete a cycle; choose MR as the next shortest.

The four edges now chosen form a spanning tree of total length 24 miles, and this is the solution.

Example

The second problem, for which the distance table is repeated overleaf, can be solved using the same algorithm. You may find it helpful to draw a diagram and mark on the edges as they are chosen. A computer could not do this, of course.

How does a computer identify (and avoid) cycles in applying this algorithm?

	T	A	B	C	D	E	F	G	H
T	-	120	150	140	120	100	160	70	180
A	120	-	60	60	90	190	210	160	40
B	150	60	-	20	80	180	170	160	50
C	140	60	20	-	40	160	150	140	60
D	120	90	80	40	-	130	70	110	120
E	100	190	180	160	130	-	140	30	220
F	160	210	170	150	70	140	-	150	200
G	70	160	160	140	110	30	150	-	200
H	180	40	50	60	120	220	200	200	-

The shortest edge is BC (20), so choose BC.

The shortest remaining is EG (30), so choose EG.

The shortest remaining is AH (40) or CD (40), so choose (say) AH at random.

The shortest remaining is CD (40), so choose CD.

The shortest remaining is BH (50), so choose BH.

The shortest remaining are AB (60), AC (60) and CH (60), but any of these would complete a cycle; the next shortest is TG (70) or FD (70), so choose TG at random.

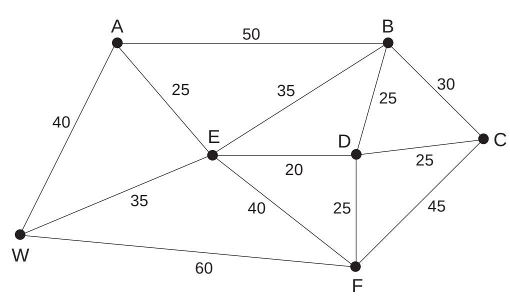
The shortest remaining (apart from those already excluded) is FD (70), so choose FD.

The shortest remaining is BD (80) but this completes a cycle, as does AD (90); the next shortest is DG (110), so choose DG.

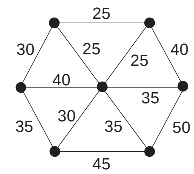
There are nine vertices altogether, so the eight edges now chosen form a minimum-length spanning tree with total length 430 km.

Exercise 2B

1. The owner of a caravan site has caravans positioned as shown in the diagram, with distances in metres between them, and wants to lay on a water supply to each of them. Use Kruskal's algorithm to determine how the caravans should be connected so that the total length of pipe required is a minimum.



2. The warden of an outdoor studies centre wants to set up a public address system linking all the huts. The distances in metres between the huts are shown in the diagram opposite. How should the huts be linked to minimise the total distance?



3. A complicated business document, currently written in English, is to be translated into each of the other European Community languages. Because it is harder to find translators for some languages than for others, some translations are more expensive than others; the costs in ECU are as shown in the table opposite.

   Use Kruskal's algorithm to decide which translations should be made so as to obtain a version in each language at minimum total cost.

From / To	Dan	Dut	Eng	Fre	Ger	Gre	Ita	Por	Spa
Danish	-	90	100	120	60	160	120	140	120
Dutch	90	-	70	80	50	130	90	120	80
English	100	70	-	50	60	150	110	150	90
French	120	80	50	-	70	120	70	100	60
German	60	50	60	70	-	120	80	130	80
Greek	160	130	150	120	120	-	100	170	150
Italian	120	90	110	70	80	100	-	110	70
Portuguese	140	120	150	100	130	170	110	-	50
Spanish	120	80	90	60	80	150	70	50	-

2.4 Prim's algorithm

Although Kruskal's algorithm is effective and fairly simple, it does create the need to check for cycles at each stage. This is easy enough when calculations are being done 'by hand' from a graph, but (as you may have discovered) is less easy when working from a table and is quite difficult to build into a computer program.

An alternative algorithm which is marginally harder to set out, but which overcomes this difficulty, is **Prim's algorithm**. Unlike Kruskal's algorithm, which looks for short edges all over the graph, Prim's algorithm starts at one vertex and builds up the spanning tree gradually from there.

The algorithm is as follows:

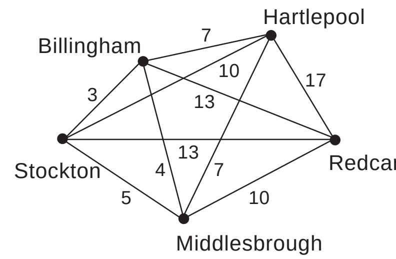
1. Start with any vertex chosen at random, and consider this as a tree.

2. Look for the shortest edge which joins a vertex on the tree to a vertex not on the tree, and add this to the tree. (If there is more than one such edge, choose any one of them at random.)

3. Repeat Step 2 until all the vertices of the graph are on the tree; the tree is then a minimum-length spanning tree.

Example

Consider once again the cable television problem from Section 2.2.



For no particular reason, choose Hartlepool as the starting point.

The shortest edge joining {H} to {B, S, M, R} is HB or HM, so choose either of these - HM, say - and add it to the tree.

The shortest edge joining {H, M} to {B, S, R} is MB, so add that to the tree.

The shortest edge joining {H, M, B} to {S, R} is BS, so add that to the tree.

The shortest edge joining {H, M, B, S} to {R} is MR, so add that to the tree.

All the vertices are now on the tree, so it is a spanning tree which (according to the algorithm) is of minimum length.

Example

Similarly, consider again the problem about the oil wells at distances as in the table:

	T	A	B	C	D	E	F	G	H
T	-	120	150	140	120	100	160	70	180
A	120	-	60	60	90	190	210	160	40
B	150	60	-	20	80	180	170	160	50
C	140	60	20	-	40	160	150	140	60
D	120	90	80	40	-	130	70	110	120
E	100	190	180	160	130	-	140	30	220
F	160	210	170	150	70	140	-	150	200
G	70	160	160	140	110	30	150	-	200
H	180	40	50	60	120	220	200	200	-

The obvious place to start is the terminal, so place T on the tree.

The shortest edge between {T} and {A, B, C, D, E, F, G, H} is TG, so add this to the tree.

The shortest edge between {T, G} and {A, B, C, D, E, F, H} is GE, so add this.

The shortest edge between {T, G, E} and {A, B, C, D, F, H} is GD.

The shortest edge between {T, G, E, D} and {A, B, C, F, H} is DC.

The shortest edge between {T, G, E, D, C} and {A, B, F, H} is CB.

The shortest edge between {T, G, E, D, C, B} and {A, F, H} is BH.

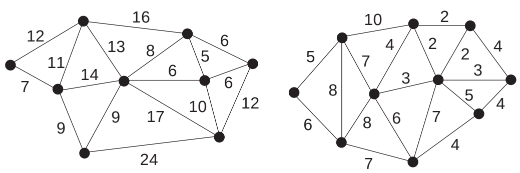
The shortest edge between {T, G, E, D, C, B, H} and {A, F} is HA.

The shortest edge between {T, G, E, D, C, B, H, A} and {F} is DF.

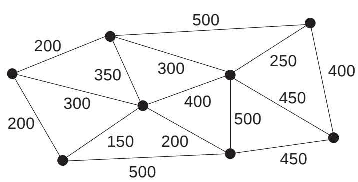
The tree now includes all the vertices, and thus is the minimum-length spanning tree required.

Exercise 2C

1. Use Prim's algorithm to solve the minimum connector problem for each of the graphs below.



2. A group of friends wants to set up a message system so that any one of them can communicate with any of the others either directly or via others in the group. If their homes are situated as shown below, with distances in metres between them as marked, use Prim's algorithm to decide where they should make the links so that the total length of the system will be as small as possible.



3. The chief greenkeeper of a nine-hole golf course plans to install an automatic sprinkler system using water from the mains at the clubhouse to water the greens. The distances in metres between the greens are as shown in the table below. Use Prim's algorithm to decide where the pipes should be installed to make their total length a minimum.

	CH	1	2	3	4	5	6	7	8
1	250								
2	600	350							
3	400	50	300						
4	100	200	600	300					
5	500	350	200	150	400				
6	800	550	200	500	750	350			
7	600	400	150	300	550	250	150		
8	350	200	350	100	350	50	400	250	
9	50	300	600	400	100	500	850	600	350

[If you want further examples for practice, repeat Exercise 2B using Prim's algorithm instead of Kruskal's algorithm.]

2.5 The travelling salesman problem

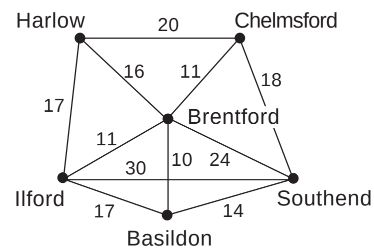
A problem not too dissimilar from the minimum connector problem concerns a travelling salesman who has to visit each of a number of towns before returning to his base. For obvious reasons the salesman wants to take the shortest available route between the towns, and the problem is simply to identify such a route.

This can be treated as a problem of finding a minimum-length Hamiltonian cycle, even though in practice the salesman has slightly more flexibility. A Hamiltonian cycle visits every vertex exactly once, you may recall, but there is nothing to stop the salesman passing through the same town twice - even along the same road twice - if that happens to provide the shortest route. This minor difference is rarely of any consequence, however, and for the purpose of this chapter it is ignored.

Activity 6

A salesman based in Harlow in Essex has to visit each of five other towns before returning to his base. If the distances in miles between the towns are as shown on the diagram opposite, in what order should he make his visits so that his total travelling distance is a minimum?

Find a solution by any method, and compare your answer with those of other students.



Southern Essex

The shortest route is actually a Hamiltonian cycle in this instance, and has a total length of 90 miles. It can be solved intuitively: there is a fairly obvious 'natural' route around the five outer towns, and all that remains is to decide when to detour via Brentford. For six towns arranged as these are, such a method is generally good enough.

Computer search

An alternative method, impractical for pencil-and-paper calculation but not unreasonable for a computer, is simply to find the total length of every possible route in turn. Given that the route must start and finish at Harlow, there are only 120 different orders in which to visit the other five towns (assuming than no town is visited more than once) and the necessary calculations can be completed in no more than a minute or two. Only half the routes really need to be checked, since the other half are the same routes in reverse, but if the time is available it is much simpler to check them all.

Take a moment or two to think about the procedure that might be used to take the computer through the 120 different routes.

The most obvious way is to keep the first town constant while the computer runs through the 24 possibilities for the other four (which it does by keeping the first of them constant while it goes through the 6 possibilities for the other three, etc.), doing this for each of the five possible first towns. A clever programmer could reduce such a scheme to just a few lines of code using recursive functions - that is, functions defined in terms of themselves!

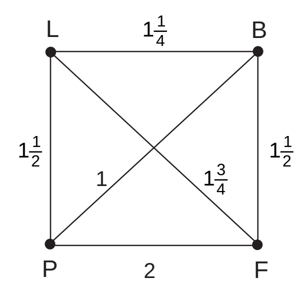
A less obvious alternative approach is by analogy with bellringing. Traditional English church bellringers do not play tunes, but instead ring 'changes' by ringing the bells in every possible order - essentially the same as the problem here. Many different 'systems' have been devised for ringing changes, with names such as 'Plain Bob', 'Steadman' and 'Grandsire', and any of these can be converted without too much difficulty to a computer program.

Now it is clear that if a computer can be used to check every possible route and find the shortest, that route will be the solution to the salesman's problem. That might appear to be an end to the matter, but in fact it is not. For six towns there are only 120 different routes to try, and that is quite manageable even on a desktop computer, but as the number of towns increases the number of possible routes increases factorially. Thus for ten towns there are 362880 possible routes, for fifteen there are nearly a hundred billion, and for twenty there are more than 10^{17}. Even the fastest computers would take many years, if not many centuries, to check all the possible routes around twenty towns, making the full search method of little use in practice.

Further examples

Example

A business executive based in London has to visit Paris, Brussels and Frankfurt before returning to London. If the journey times in hours are as shown in the diagram opposite, work out the total length of every possible Hamiltonian cycle and thus find the route that takes the shortest time.



Journey times

There are three cities apart from London, and so $3 \times 2 \times 1 = 6$ possible orders. The six journeys with their total lengths are as follows:

L - P - B - F - L = $5\frac{3}{4}$ h   L - F - B - P - L = $5\frac{3}{4}$ h

L - P - F - B - L = $6\frac{1}{4}$ h   L - B - F - P - L = $6\frac{1}{4}$ h

L - B - P - F - L = 6 h   L - F - P - B - L = 6 h

It is clear from this that the shortest route is

London - Paris - Brussels - Frankfurt - London

or the same in reverse.

Example

A visitor to the County Show wants to start from the main gate, visit each of eight exhibitions, and return to the main gate by the shortest possible route. The distances in metres between the exhibitions are given in the table below. What route should the visitor take?

	Gate	A	B	C	D	E	F	G	H
Gate	-	200	350	400	500	350	150	200	350
A	200	-	200	300	400	450	300	250	200
B	350	200	-	100	250	450	500	300	100
C	400	300	100	-	150	350	500	300	100
D	500	400	250	150	-	250	450	300	200
E	350	450	450	350	250	-	250	200	350
F	150	300	500	500	450	250	-	200	400
G	200	250	300	300	300	200	200	-	200
H	350	200	100	100	200	350	400	200	-

Without a graph this problem is very difficult, but it is possible to make some sort of attempt at a solution.

Activity 7

Try to find a minimum-length route and compare your answer with those of other students.

The 'best' solution has a total length of 1550 m. This is the length of the route Gate - A - H - B - C - D - E - G - F - Gate, but there are other routes of the same minimum length. If you found any of these routes for yourself, you should feel quite pleased.

You might expect at this point to be given a standard algorithm for the solution of the travelling salesman problem. Unfortunately, no workable general algorithm has yet been discovered - the exhaustive search method is reliable but (for large vertex sets) takes too long to be of any practical use. Intuitive 'common sense' methods can often lead to the best (or nearly best) solution in particular cases, but the general problem has yet to be solved.

* Upper and lower bounds

Although there is no general algorithm for the solution of the travelling salesman problem, it is possible to find upper and lower bounds for the minimum distance required. This can sometimes be very useful, because if you know that the shortest route is between (say) 47 miles and 55 miles long, and you can find a route of length 47 miles, you know that your answer is actually a solution. Alternatively, from a business point of view, if the best route you can find by trial-and-error is 48 miles long, you might well decide that the expense of looking for a shorter route was just not worthwhile.

Finding an upper bound is easy: simply work out the length of any Hamiltonian cycle. Since this cycle gives a possible solution, the best solution must be no longer than this length. If the graph is not too different from an ordinary map drawn to scale, it is usually possible by a sensible choice of route to find an upper bound quite close to the minimum length.

Finding a good lower bound is a little trickier, but not impossible. Suppose that in a graph of 26 vertices, A - Z, you had a minimum length Hamiltonian cycle, AB, BC, CD, ..., YZ, ZA. If you remove any one of its vertices, Z say, and look at the remaining graph on vertices A - Y then

minimum length of a Hamiltonian cycle

$$= \text{length of AB, BC, CD, ..., XY, YZ, ZA}$$

$$= (\text{length of AB, BC, CD, ..., XY}) + (\text{length of YZ}) + (\text{length of ZA})$$

$$\geq \begin{pmatrix}\text{minimum length of a spanning tree} \\ \text{of the graph on vertices A - Y}\end{pmatrix} + \begin{pmatrix}\text{lengths of the two} \\ \text{shortest edges from Z}\end{pmatrix}$$

So a lower bound for the minimum-length Hamiltonian cycle is given by the minimum length of a spanning tree of the original graph without one of its vertices, added to the lengths of the two shortest edges from the remaining vertex.

Look at the County Show problem on pages 29/30 and see how these two methods work. An upper bound is easy to find: the cycle

Gate -A - B -C- D - E - F - G - H - Gate

has total length 1900 m, so the optimum solution must be no more than this. Simply by trial and error, you may be able to find a shorter route giving a better upper bound.

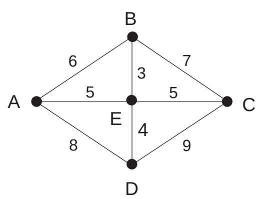
Now let us consider the problem of finding a good lower bound by the method just given. Suppose the vertex E is removed. Using Prim's algorithm as in Section 2.4, the minimum-length spanning tree for the remaining vertices has length 1100 m. The two shortest edges from E are 200 and 250, and adding these to the spanning tree for the rest gives 1550 m. The optimum solution must have at least this length, but may in fact be longer - there is no certainty that a route as short as this exists. The best route around the Show thus has a length between 1550 m and 1900 m.

This technique gives no indication as to how such a best route can be found, of course, although a minimum-length spanning tree may be a useful starter. But if by clever guesswork you can find a route equal in length to the lower bound, you can be certain that it is in fact a minimum-length route. The route Gate - A - H - B - C - D - E - G - F - Gate is an example: it has length 1550 m, equal to the lower bound, and so is certainly a minimum.

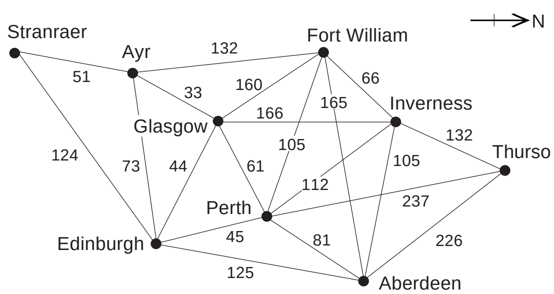
Exercise 2D

The following problems may be solved by systematic search by hand or computer, by finding upper and lower bounds, by trial and error, or in any other way.

1. Find a minimum-length Hamiltonian cycle on the graph shown in the diagram below.



2. The Director of the Scottish Tourist Board, based in Edinburgh, plans a tour of inspection around each of her District Offices, finishing back at her own base. The distances in miles between the offices are shown in the diagram; find a suitable route of minimum length.



3. A milling machine can produce four different types of component as long as its settings are changed for each type. The times in minutes required to change settings are shown in the table.

From / To	A	B	C	D	Off
A	-	5	7	4	8
B	5	-	6	7	6
C	7	6	-	5	9
D	4	7	5	-	7
Off	8	6	9	7	-

On a particular day, some of each component have to be produced. If the machine must start and finish at 'Off', find the order in which the components should be made so that the time wasted in changing settings is as low as possible.

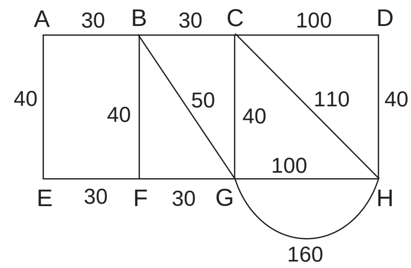
2.6 The Chinese postman problem

The Chinese postman problem takes its name not from the postman's nationality, but from the fact that it was first seriously studied by the Chinese mathematician *Mei-ko Kwan* in the 1960s. It concerns a postman who has to deliver mail to houses along each of the streets in a particular housing estate, and wants to minimise the distance he has to walk.

How does this differ from the problem of the travelling salesman?

The travelling salesman wants to visit each town - each vertex, to use the language of graph theory - so the solution is a Hamiltonian cycle. The postman, on the other hand, wants to travel along each road - each edge of the graph - so his problem can best be solved by an Eulerian trail if such a thing exists. Most graphs are not Eulerian, however, and this is what makes the problem interesting.

The diagram shows a housing estate with the length in metres of each road. The postman's round must begin and end at A, and must take him along each section of road at least once.



Activity 8

Try to find a route of minimum total length.

Once again, trial and error will often lead to a satisfactory solution where the number of vertices is small, but for larger graphs an algorithm is desirable. A complete algorithm for the solution of the Chinese postman problem does exist, but it is too complicated to set out in full here. What follows is a much simpler partial algorithm that will work reasonably well in most cases.

Systematic solution

The algorithm combines the idea of an Eulerian trail with that of a shortest path. You will recall that an Eulerian graph can be identified by the fact that all its vertices have even degree, and this is at the heart of the systematic solution. The method is as follows:

Find the degree of each vertex of the graph.

1. If all the vertices are of even degree, the graph is Eulerian and any Eulerian trail is an acceptable shortest route.

2. If just two vertices are of odd degree, use the algorithm from Section 2.1 to find the shortest path between them; the postman must walk these edges twice and each of the others once.

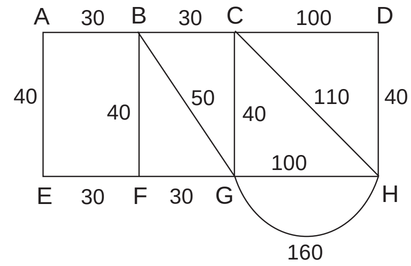
3. If more than two vertices are of odd degree - this is where the partial algorithm fails - use common sense to look for the shortest combination of paths between pairs of them. These are the edges that the postman must walk twice.

Example

Consider again the problem above (regarding the layout of roads as a graph with vertices A - H). The only two odd vertices are F and G, and the shortest path between them is obviously the direct edge FG. The postman must therefore walk this section of road twice and all the rest once: a possible route would be

A - B - C - D - H - G - H - C - G - B - F - G - F - E - A.

The total length of any such shortest route is the sum of all the edge lengths plus the repeated edge, which is 830 m.



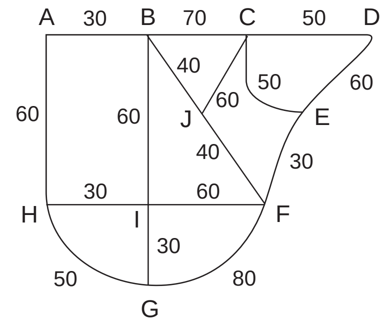
Example

A roadsweeper has to cover the road system shown in the diagram opposite, going along every road at least once but travelling no further than necessary. What route should it take?

There are four odd vertices, E, G, H and J. By common sense, the shortest combined path comes from joining H to G directly, and J to E via F, so the repeated edges must be HG, FJ and FE with a combined length of 120 m. The edges of the graph have total length 800 m, so the sweeper's best route is 920 m long. One such route is

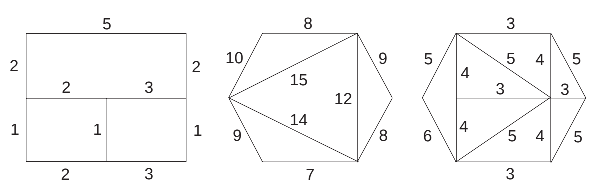
A - B - C - D - E - C - J - F - E - F - I - B - J - F - G - I - H - G - H - A

but there are many other equally valid routes.

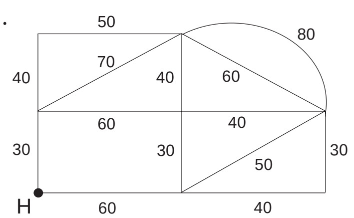


Exercise 2E

1. Find a solution to the Chinese postman problem on each of the graphs below.

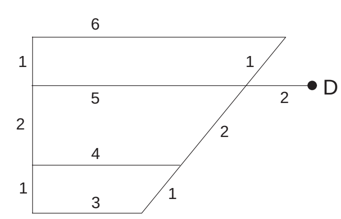


2. A church member has to deliver notices to the houses along each of the roads shown in the diagram below. The distances shown are in metres.



If her own house is at H, what route should she follow in order to make her total walking distance as short as possible?

3. After a night of heavy snow, the County Council sends out its snow plough to clear the main roads shown (with their lengths in km) in the diagram below.



The plough must drive at least once along each of the roads to clear it, but should obviously take the shortest route starting and finishing at the depot D. Which way should it go?

2.7 Local applications

The previous sections of this chapter have covered four classes of problem: the shortest path problem, the minimum connector problem, the travelling salesman problem and the Chinese postman problem. They have some similarities, but each class of problem requires a slightly different method for its solution, and it is important to recognise the different problems when they occur.

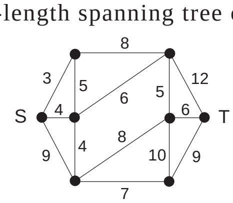
One way of acquiring the ability to distinguish the four problems from one another is to consider a selection of real life problems and try to classify them - even try to solve them, if they are not too complicated. Time set aside for this purpose will not be wasted.

The problems you find will depend on your own local environment, but might include

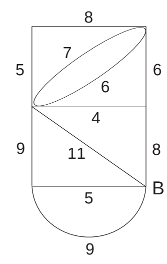
- finding the quickest way of getting from one point in the city to another, on foot, or on a cycle, or by car, or by bus;
- finding the shortest route around your school or college if you have a message to deliver to every classroom;
- finding the shortest route that takes you along every line of your nearest Underground or Metro system, and trying it out in practice;
- planning a hike or expedition visiting each of six pre-chosen churches, or hilltops, or other schools, or pubs;
- whatever catches your imagination.

2.8 Miscellaneous Exercises

1. Use Kruskal's or Prim's algorithm to find a minimum-length spanning tree on the graph below.



2. Use the algorithm given to find the shortest path from S to T on the graph above.
3. Solve the Chinese postman problem for the graph shown in the diagram above.
4. Find a minimum-length Hamiltonian cycle on the graph above.
5. A railway track inspector wishes to inspect all the tracks shown in the diagram below, starting and finishing at the base B (distances shown in km).



Explain why this cannot be done without going over some sections of track more than once, and find the shortest route the inspector can take.

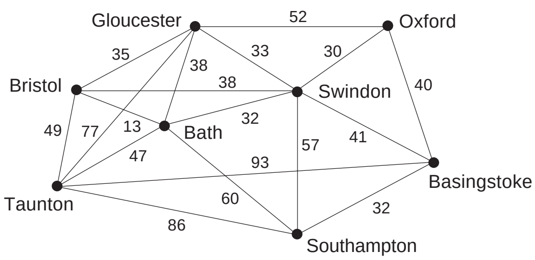
6. A building society with offices throughout Avon wants to link its branches in a private computer network.

The distances in miles between the branches are as shown in the table opposite.

Find a way of connecting them so that the total length of cable required is a minimum.

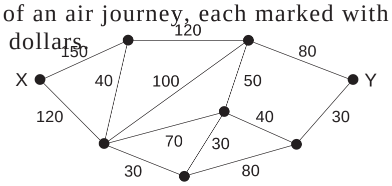
	BA	BR	CS	CL	KE	KI	LU	PA	PO	RA	WE
Bath		13	12	23	7	9	16	16	22	8	29
Bristol			11	12	6	5	8	7	9	14	21
Chipping Sodbury				23	11	7	19	8	19	20	32
Clevedon					17	17	8	16	5	22	8
Keynsham						4	10	11	15	9	23
Kingswood							13	7	14	13	26
Lulsgate								15	8	14	13
Patchway									11	20	24
Portishead										21	13
Radstock											24
Weston Super Mare											

7. A sales rep based in Bristol has to visit shops in each of seven other towns before returning to her base. The distances in miles between the towns are as shown in the diagram below.



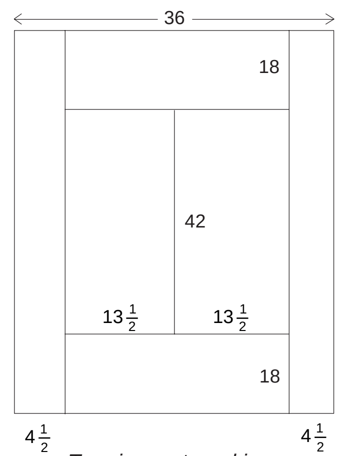
Find a suitable route of minimum length.

8. The diagram below shows the various possible stages of an air journey, each marked with its cost in dollars.



Use a suitable algorithm to find the least expensive route from X to Y, and state its cost.

9. The groundsman of a tennis club has to mark out the court with white lines, with distances in feet as shown in the diagram. Because the painting machine is faulty, it cannot be turned off and so must go only along the lines to be marked. How far must the groundsman walk, given that he need not finish at the same place he started?



Tennis court markings

	L	O	C	S	G	E	B	N	H
London	-	$1\frac{1}{2}$	$1\frac{1}{2}$	3	5	$4\frac{1}{2}$	2	3	1
Oxford	$1\frac{1}{2}$	-	3	$1\frac{1}{2}$	6	6	2	$4\frac{1}{2}$	$1\frac{1}{2}$
Cambridge	$1\frac{1}{2}$	3	-	4	6	$5\frac{1}{2}$	$3\frac{1}{2}$	2	$2\frac{1}{2}$
Stratford	3	$1\frac{1}{2}$	4	-	5	$6\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	3
Grasmere	5	6	6	5	-	4	7	8	6
Edinburgh	$4\frac{1}{2}$	6	$5\frac{1}{2}$	$6\frac{1}{2}$	4	-	$6\frac{1}{2}$	7	$5\frac{1}{2}$
Bath	2	2	$3\frac{1}{2}$	$3\frac{1}{2}$	7	$6\frac{1}{2}$	-	5	2
Norwich	3	$4\frac{1}{2}$	2	$5\frac{1}{2}$	8	7	5	-	4
Heathrow	1	$1\frac{1}{2}$	$2\frac{1}{2}$	3	6	$5\frac{1}{2}$	2	4	-

10. An American tourist arrives at Heathrow and wants to visit London, Oxford, Cambridge, Stratford-on-Avon, Grasmere, Edinburgh, Bath, and her ancestors' home in Norwich before returning to Heathrow to catch a flight back to the USA.

The travel times in hours between places are as shown in the table opposite.

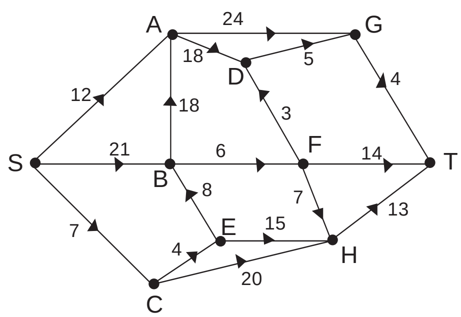
If she wants to spend 6 hours in each place, can she complete such a journey in the 75 waking hours she has available before her flight leaves?

11. The following table gives the distances between five towns, A, B, C, D and E. Use Kruskal's algorithm to construct a minimum connector joining these towns.

	A	B	C	D	E
A	-	9	3	7	6
B	9	-	8	9	8
C	3	8	-	7	7
D	7	9	7	-	4
E	6	8	7	4	-

12. Use the shortest path algorithm to find the shortest path from S to T in the following network.

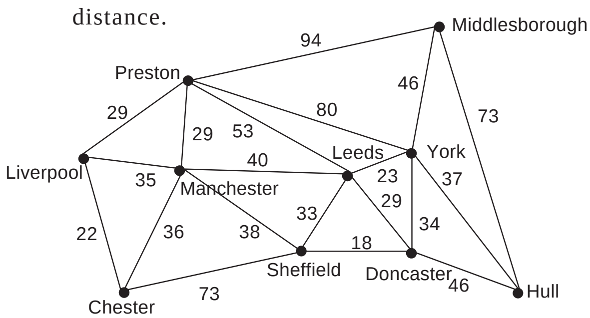
(Your answer should show clearly how the algorithm is being applied, and what the vertex-labels are at each stage.)



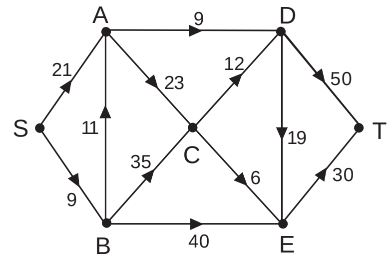
13. The network in the diagram below indicates the main road system between 10 towns and cities in the North of England.

(a) A computer company wishes to install a computer network between these places, using cables laid alongside the roads and designed so that all places are connected (either directly or indirectly) to the main computer located at Manchester. Find the network which uses a minimum total length of cable.

(b) A highways maintenance team, based at Manchester, wishes to inspect all the roads shown in the network at least once. Design the route for them, which starts and finishes at Manchester, and has the smallest total distance.



14. Use the shortest-path algorithm to find the shortest path between S and T for the following network.



The numbers represent the lengths of each arc and you can only move in the directions indicated by the arrows.

15. The following table gives the distance (in km) between six water pumping stations.

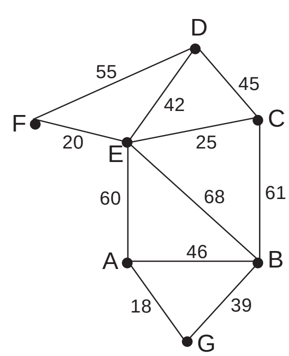
	A	B	C	D	E	F
A	-	74	64	98	83	144
B	74	-	81	145	144	161
C	64	81	-	50	100	80
D	98	145	50	-	83	63
E	83	144	100	83	-	151
F	144	161	80	63	151	-

(a) Use an algorithm to construct a minimum connector between the stations.

(b) Explain how the algorithm can be used in general to provide a lower bound for the solution of the travelling salesman problem, and obtain such a lower bound for the six pumping stations.

(c) Find an upper bound for this problem.

16. The graph below represents a railway network connecting towns A, B, C, D, E, F, and G, with distances between towns shown in kilometres.



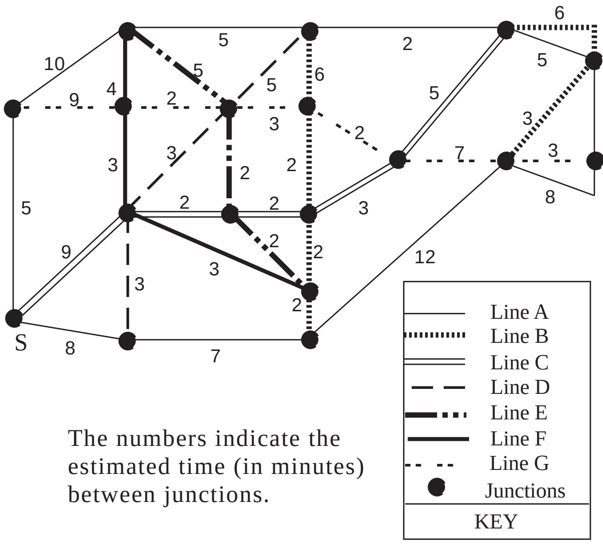
An inspection train is required to begin at B and survey all of the track and return to B. The train needs to travel only in one direction along any section of track in order to carry out the survey of that section.

(a) Explain why it is impossible to complete the survey by travelling along each section of track once and only once.

(b) Find a route of minimum length for the train, stating which section(s) of track must be repeated, and determine the total route length.

(c) Is the solution found in (b) unique? Give an explanation for your answer. (AEB)

17. The network below illustrates the main junctions on a city rail sytem.



The numbers indicate the estimated time (in minutes) between junctions.

(a) Use an appropriate shortest path algorithm to find the path of minimum total time between S and T, assuming that there is no waiting when changing trains at junctions.

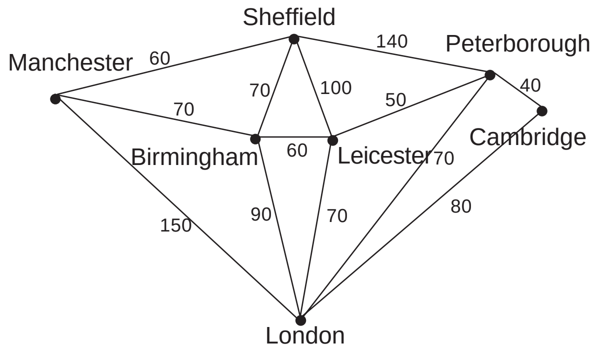
(b) Assuming that an average time for changing and waiting at a junction is 5 minutes, is the solution determined in (a) still valid? If not, what is now the optimal solution?

(Note that, at a junction, you only change trains if you are changing from one line to a different one.) (AEB)

18. The network below illustrates the time (in minutes) of rail journeys between the given cities.

(a) Assuming that any necessary changes of train do not add any time to the journey, find the quickest train route from Manchester to Cambridge.

(b) Assuming that at each city shown a change of train is always necessary, and that each change adds 20 minutes to the journey time, find the quickest train route from Manchester to Cambridge.



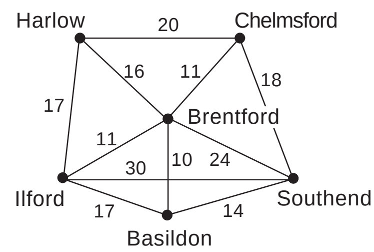
(AEB)

slightly more flexibility. A Hamiltonian cycle visits every vertex exactly once, you may recall, but there is nothing to stop the salesman passing through the same town twice - even along the same road twice - if that happens to provide the shortest route. This minor difference is rarely of any consequence, however, and for the purpose of this chapter it is ignored.

Activity 6

A salesman based in Harlow in Essex has to visit each of five other towns before returning to his base. If the distances in miles between the towns are as shown on the diagram opposite, in what order should he make his visits so that his total travelling distance is a minimum?

Find a solution by any method, and compare your answer with those of other students.



Southern Essex

The shortest route is actually a Hamiltonian cycle in this instance, and has a total length of 90 miles. It can be solved intuitively: there is a fairly obvious 'natural' route around the five outer towns, and all that remains is to decide when to detour via Brentford. For six towns arranged as these are, such a method is generally good enough.

Computer search

An alternative method, impractical for pencil-and-paper calculation but not unreasonable for a computer, is simply to find the total length of every possible route in turn. Given that the route must start and finish at Harlow, there are only 120 different orders in which to visit the other five towns (assuming than no town is visited more than once) and the necessary calculations can be completed in no more than a minute or two. Only half the routes really need to be checked, since the other half are the same routes in reverse, but if the time is available it is much simpler to check them all.

Take a moment or two to think about the procedure that might be used to take the computer through the 120 different routes.

The most obvious way is to keep the first town constant while the computer runs through the 24 possibilities for the other four (which it does by keeping the first of them constant while it goes through the 6 possibilities for the other three, etc.), doing this for each of the five possible first towns. A clever programmer could reduce such a scheme to just a few lines of code using recursive functions - that is, functions defined in terms of themselves!

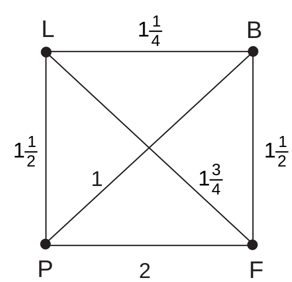
A less obvious alternative approach is by analogy with bellringing. Traditional English church bellringers do not play tunes, but instead ring 'changes' by ringing the bells in every possible order - essentially the same as the problem here. Many different 'systems' have been devised for ringing changes, with names such as 'Plain Bob', 'Steadman' and 'Grandsire', and any of these can be converted without too much difficulty to a computer program.

Now it is clear that if a computer can be used to check every possible route and find the shortest, that route will be the solution to the salesman's problem. That might appear to be an end to the matter, but in fact it is not. For six towns there are only 120 different routes to try, and that is quite manageable even on a desktop computer, but as the number of towns increases the number of possible routes increases factorially. Thus for ten towns there are 362880 possible routes, for fifteen there are nearly a hundred billion, and for twenty there are more than 10^{17}. Even the fastest computers would take many years, if not many centuries, to check all the possible routes around twenty towns, making the full search method of little use in practice.

Further examples

Example

A business executive based in London has to visit Paris, Brussels and Frankfurt before returning to London. If the journey times in hours are as shown in the diagram opposite, work out the total length of every possible Hamiltonian cycle and thus find the route that takes the shortest time.



Journey times

There are three cities apart from London, and so $3 \times 2 \times 1 = 6$ possible orders. The six journeys with their total lengths are as follows:

L - P - B - F - L = $5\frac{3}{4}$ h	L - F - B - P - L = $5\frac{3}{4}$ h
L - P - F - B - L = $6\frac{1}{4}$ h	L - B - F - P - L = $6\frac{1}{4}$ h
L - B - P - F - L = 6 h	L - F - P - B - L = 6 h

It is clear from this that the shortest route is

London - Paris - Brussels - Frankfurt - London

or the same in reverse.

Example

A visitor to the County Show wants to start from the main gate, visit each of eight exhibitions, and return to the main gate by the shortest possible route. The distances in metres between the exhibitions are given in the table below. What route should the visitor take?

	Gate	A	B	C	D	E	F	G	H
Gate	-	200	350	400	500	350	150	200	350
A	200	-	200	300	400	450	300	250	200
B	350	200	-	100	250	450	500	300	100
C	400	300	100	-	150	350	500	300	100
D	500	400	250	150	-	250	450	300	200
E	350	450	450	350	250	-	250	200	350
F	150	300	500	500	450	250	-	200	400
G	200	250	300	300	300	200	200	-	200
H	350	200	100	100	200	350	400	200	-

Without a graph this problem is very difficult, but it is possible to make some sort of attempt at a solution.

Activity 7

Try to find a minimum-length route and compare your answer with those of other students.

The 'best' solution has a total length of 1550 m. This is the length of the route Gate - A - H - B - C - D - E - G - F - Gate, but there are other routes of the same minimum length. If you found any of these routes for yourself, you should feel quite pleased.

You might expect at this point to be given a standard algorithm for the solution of the travelling salesman problem. Unfortunately, no workable general algorithm has yet been discovered - the exhaustive search method is reliable but (for large vertex sets) takes too long to be of any practical use. Intuitive 'common sense' methods can often lead to the best (or nearly best) solution in particular cases, but the general problem has yet to be solved.

* Upper and lower bounds

Although there is no general algorithm for the solution of the travelling salesman problem, it is possible to find upper and lower bounds for the minimum distance required. This can sometimes be very useful, because if you know that the shortest route is between (say) 47 miles and 55 miles long, and you can find a route of length 47 miles, you know that your answer is actually a solution. Alternatively, from a business point of view, if the best route you can find by trial-and-error is 48 miles long, you might well decide that the expense of looking for a shorter route was just not worthwhile.

Finding an upper bound is easy: simply work out the length of any Hamiltonian cycle. Since this cycle gives a possible solution, the best solution must be no longer than this length. If the graph is not too different from an ordinary map drawn to scale, it is usually possible by a sensible choice of route to find an upper bound quite close to the minimum length.

Finding a good lower bound is a little trickier, but not impossible. Suppose that in a graph of 26 vertices, A - Z, you had a minimum length Hamiltonian cycle, AB, BC, CD, ..., YZ, ZA. If you remove any one of its vertices, Z say, and look at the remaining graph on vertices A - Y then

minimum length of a Hamiltonian cycle

= length of AB, BC, CD, ..., XY, YZ, ZA

= (length of AB, BC, CD, ..., XY) + (length of YZ) + (length of ZA)

$$\geq \begin{pmatrix} \text{minimum length of a spanning tree} \\ \text{of the graph on vertices A - Y} \end{pmatrix} + \begin{pmatrix} \text{lengths of the two} \\ \text{shortest edges from Z} \end{pmatrix}$$

So a lower bound for the minimum-length Hamiltonian cycle is given by the minimum length of a spanning tree of the original graph without one of its vertices, added to the lengths of the two shortest edges from the remaining vertex.

Look at the County Show problem on pages 29/30 and see how these two methods work. An upper bound is easy to find: the cycle

Gate -A - B -C- D - E - F - G - H - Gate

has total length 1900 m, so the optimum solution must be no more than this. Simply by trial and error, you may be able to find a shorter route giving a better upper bound.

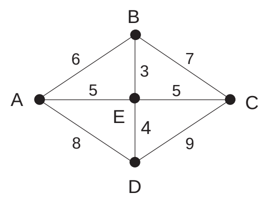
Now let us consider the problem of finding a good lower bound by the method just given. Suppose the vertex E is removed. Using Prim's algorithm as in Section 2.4, the minimum-length spanning tree for the remaining vertices has length 1100 m. The two shortest edges from E are 200 and 250, and adding these to the spanning tree for the rest gives 1550 m. The optimum solution must have at least this length, but may in fact be longer - there is no certainty that a route as short as this exists. The best route around the Show thus has a length between 1550 m and 1900 m.

This technique gives no indication as to how such a best route can be found, of course, although a minimum-length spanning tree may be a useful starter. But if by clever guesswork you can find a route equal in length to the lower bound, you can be certain that it is in fact a minimum-length route. The route Gate - A - H - B - C - D - E - G - F - Gate is an example: it has length 1550 m, equal to the lower bound, and so is certainly a minimum.

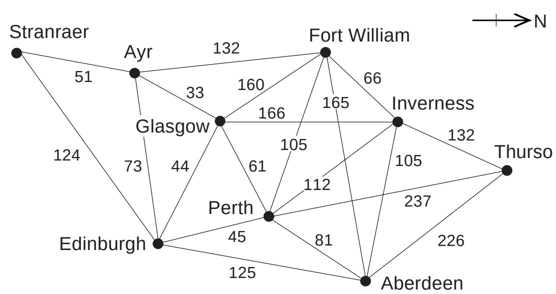
Exercise 2D

The following problems may be solved by systematic search by hand or computer, by finding upper and lower bounds, by trial and error, or in any other way.

1. Find a minimum-length Hamiltonian cycle on the graph shown in the diagram below.



2. The Director of the Scottish Tourist Board, based in Edinburgh, plans a tour of inspection around each of her District Offices, finishing back at her own base. The distances in miles between the offices are shown in the diagram; find a suitable route of minimum length.



3. A milling machine can produce four different types of component as long as its settings are changed for each type. The times in minutes required to change settings are shown in the table.

From / To	A	B	C	D	Off
A	-	5	7	4	8
B	5	-	6	7	6
C	7	6	-	5	9
D	4	7	5	-	7
Off	8	6	9	7	-

On a particular day, some of each component have to be produced. If the machine must start and finish at 'Off', find the order in which the components should be made so that the time wasted in changing settings is as low as possible.

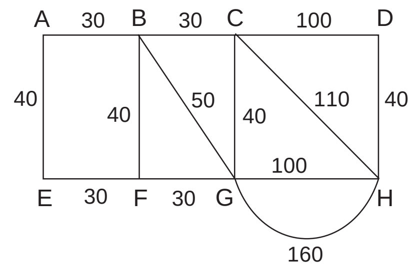
2.6 The Chinese postman problem

The Chinese postman problem takes its name not from the postman's nationality, but from the fact that it was first seriously studied by the Chinese mathematician *Mei-ko Kwan* in the 1960s. It concerns a postman who has to deliver mail to houses along each of the streets in a particular housing estate, and wants to minimise the distance he has to walk.

How does this differ from the problem of the travelling salesman?

The travelling salesman wants to visit each town - each vertex, to use the language of graph theory - so the solution is a Hamiltonian cycle. The postman, on the other hand, wants to travel along each road - each edge of the graph - so his problem can best be solved by an Eulerian trail if such a thing exists. Most graphs are not Eulerian, however, and this is what makes the problem interesting.

The diagram shows a housing estate with the length in metres of each road. The postman's round must begin and end at A, and must take him along each section of road at least once.



Activity 8

Try to find a route of minimum total length.

Once again, trial and error will often lead to a satisfactory solution where the number of vertices is small, but for larger graphs an algorithm is desirable. A complete algorithm for the solution of the Chinese postman problem does exist, but it is too complicated to set out in full here. What follows is a much simpler partial algorithm that will work reasonably well in most cases.

Systematic solution

The algorithm combines the idea of an Eulerian trail with that of a shortest path. You will recall that an Eulerian graph can be identified by the fact that all its vertices have even degree, and this is at the heart of the systematic solution. The method is as follows:

Find the degree of each vertex of the graph.

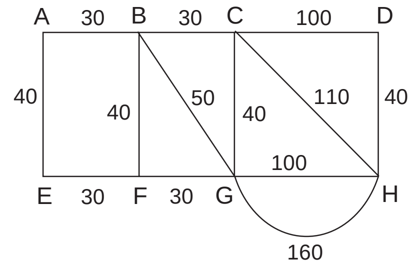
1. If all the vertices are of even degree, the graph is Eulerian and any Eulerian trail is an acceptable shortest route.

2. If just two vertices are of odd degree, use the algorithm from Section 2.1 to find the shortest path between them; the postman must walk these edges twice and each of the others once.

3. If more than two vertices are of odd degree - this is where the partial algorithm fails - use common sense to look for the shortest combination of paths between pairs of them. These are the edges that the postman must walk twice.

Example

Consider again the problem above (regarding the layout of roads as a graph with vertices A - H). The only two odd vertices are F and G, and the shortest path between them is obviously the direct edge FG. The postman must therefore walk this section of road twice and all the rest once: a possible route would be

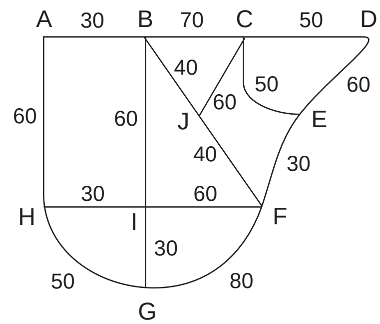


A - B - C - D - H - G - H - C - G - B - F - G - F - E - A.

The total length of any such shortest route is the sum of all the edge lengths plus the repeated edge, which is 830 m.

Example

A roadsweeper has to cover the road system shown in the diagram opposite, going along every road at least once but travelling no further than necessary. What route should it take?



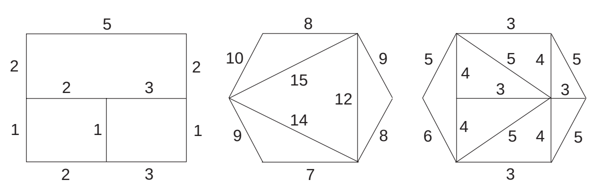
There are four odd vertices, E, G, H and J. By common sense, the shortest combined path comes from joining H to G directly, and J to E via F, so the repeated edges must be HG, FJ and FE with a combined length of 120 m. The edges of the graph have total length 800 m, so the sweeper's best route is 920 m long. One such route is

A - B - C - D - E - C - J - F - E - F - I - B - J - F - G - I - H - G - H - A

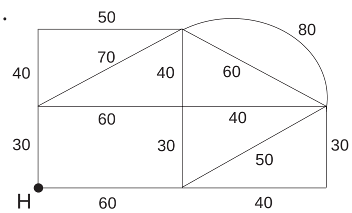
but there are many other equally valid routes.

Exercise 2E

1. Find a solution to the Chinese postman problem on each of the graphs below.

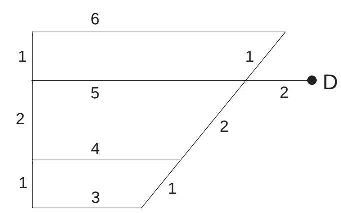


2. A church member has to deliver notices to the houses along each of the roads shown in the diagram below. The distances shown are in metres.



If her own house is at H, what route should she follow in order to make her total walking distance as short as possible?

3. After a night of heavy snow, the County Council sends out its snow plough to clear the main roads shown (with their lengths in km) in the diagram below.



The plough must drive at least once along each of the roads to clear it, but should obviously take the shortest route starting and finishing at the depot D. Which way should it go?

2.7 Local applications

The previous sections of this chapter have covered four classes of problem: the shortest path problem, the minimum connector problem, the travelling salesman problem and the Chinese postman problem. They have some similarities, but each class of problem requires a slightly different method for its solution, and it is important to recognise the different problems when they occur.

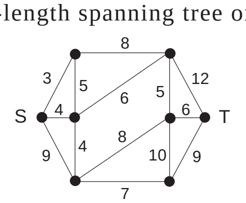
One way of acquiring the ability to distinguish the four problems from one another is to consider a selection of real life problems and try to classify them - even try to solve them, if they are not too complicated. Time set aside for this purpose will not be wasted.

The problems you find will depend on your own local environment, but might include

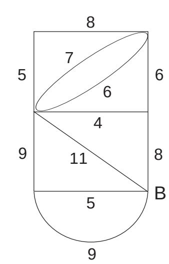
- finding the quickest way of getting from one point in the city to another, on foot, or on a cycle, or by car, or by bus;
- finding the shortest route around your school or college if you have a message to deliver to every classroom;
- finding the shortest route that takes you along every line of your nearest Underground or Metro system, and trying it out in practice;
- planning a hike or expedition visiting each of six pre-chosen churches, or hilltops, or other schools, or pubs;
- whatever catches your imagination.

2.8 Miscellaneous Exercises

1. Use Kruskal's or Prim's algorithm to find a minimum-length spanning tree on the graph below.



2. Use the algorithm given to find the shortest path from S to T on the graph above.
3. Solve the Chinese postman problem for the graph shown in the diagram above.
4. Find a minimum-length Hamiltonian cycle on the graph above.
5. A railway track inspector wishes to inspect all the tracks shown in the diagram below, starting and finishing at the base B (distances shown in km).



Explain why this cannot be done without going over some sections of track more than once, and find the shortest route the inspector can take.

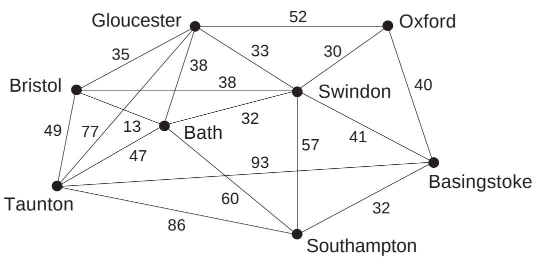
6. A building society with offices throughout Avon wants to link its branches in a private computer network.

   The distances in miles between the branches are as shown in the table opposite.

   Find a way of connecting them so that the total length of cable required is a minimum.

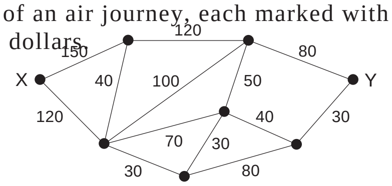
	BA	BR	CS	CL	KE	KI	LU	PA	PO	RA	WE
Bath		13	12	23	7	9	16	16	22	8	29
Bristol			11	12	6	5	8	7	9	14	21
Chipping Sodbury				23	11	7	19	8	19	20	32
Clevedon					17	17	8	16	5	22	8
Keynsham						4	10	11	15	9	23
Kingswood							13	7	14	13	26
Lulsgate								15	8	14	13
Patchway									11	20	24
Portishead										21	13
Radstock											24
Weston Super Mare											

7. A sales rep based in Bristol has to visit shops in each of seven other towns before returning to her base. The distances in miles between the towns are as shown in the diagram below.



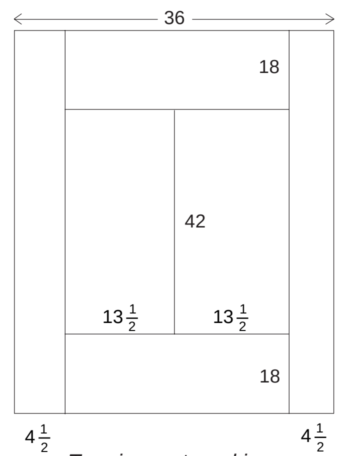
   Find a suitable route of minimum length.

8. The diagram below shows the various possible stages of an air journey, each marked with its cost in dollars.



   Use a suitable algorithm to find the least expensive route from X to Y, and state its cost.

9. The groundsman of a tennis club has to mark out the court with white lines, with distances in feet as shown in the diagram. Because the painting machine is faulty, it cannot be turned off and so must go only along the lines to be marked. How far must the groundsman walk, given that he need not finish at the same place he started?



Tennis court markings

10. An American tourist arrives at Heathrow and wants to visit London, Oxford, Cambridge, Stratford-on-Avon, Grasmere, Edinburgh, Bath, and her ancestors' home in Norwich before returning to Heathrow to catch a flight back to the USA.

    The travel times in hours between places are as shown in the table opposite.

    If she wants to spend 6 hours in each place, can she complete such a journey in the 75 waking hours she has available before her flight leaves?

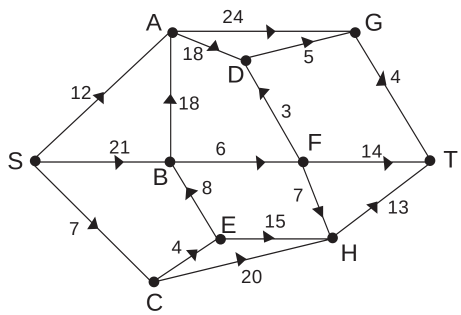
	L	O	C	S	G	E	B	N	H
London	-	$1\frac{1}{2}$	$1\frac{1}{2}$	3	5	$4\frac{1}{2}$	2	3	1
Oxford	$1\frac{1}{2}$	-	3	$1\frac{1}{2}$	6	6	2	$4\frac{1}{2}$	$1\frac{1}{2}$
Cambridge	$1\frac{1}{2}$	3	-	4	6	$5\frac{1}{2}$	$3\frac{1}{2}$	2	$2\frac{1}{2}$
Stratford	3	$1\frac{1}{2}$	4	-	5	$6\frac{1}{2}$	$3\frac{1}{2}$	$5\frac{1}{2}$	3
Grasmere	5	6	6	5	-	4	7	8	6
Edinburgh	$4\frac{1}{2}$	6	$5\frac{1}{2}$	$6\frac{1}{2}$	4	-	$6\frac{1}{2}$	7	$5\frac{1}{2}$
Bath	2	2	$3\frac{1}{2}$	$3\frac{1}{2}$	7	$6\frac{1}{2}$	-	5	2
Norwich	3	$4\frac{1}{2}$	2	$5\frac{1}{2}$	8	7	5	-	4
Heathrow	1	$1\frac{1}{2}$	$2\frac{1}{2}$	3	6	$5\frac{1}{2}$	2	4	-

11. The following table gives the distances between five towns, A, B, C, D and E. Use Kruskal's algorithm to construct a minimum connector joining these towns.

	A	B	C	D	E
A	-	9	3	7	6
B	9	-	8	9	8
C	3	8	-	7	7
D	7	9	7	-	4
E	6	8	7	4	-

12. Use the shortest path algorithm to find the shortest path from S to T in the following network.

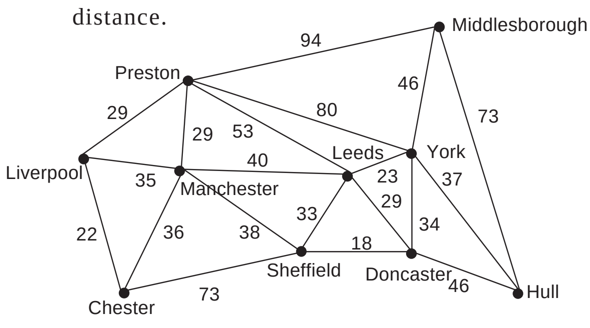
(Your answer should show clearly how the algorithm is being applied, and what the vertex-labels are at each stage.)



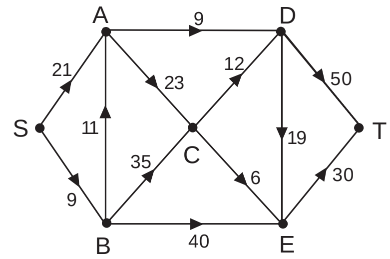
13. The network in the diagram below indicates the main road system between 10 towns and cities in the North of England.

(a) A computer company wishes to install a computer network between these places, using cables laid alongside the roads and designed so that all places are connected (either directly or indirectly) to the main computer located at Manchester. Find the network which uses a minimum total length of cable.

(b) A highways maintenance team, based at Manchester, wishes to inspect all the roads shown in the network at least once. Design the route for them, which starts and finishes at Manchester, and has the smallest total distance.



14. Use the shortest-path algorithm to find the shortest path between S and T for the following network.



The numbers represent the lengths of each arc and you can only move in the directions indicated by the arrows.

15. The following table gives the distance (in km) between six water pumping stations.

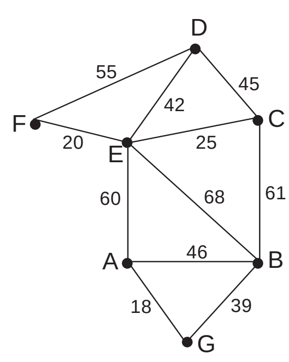
	A	B	C	D	E	F
A	-	74	64	98	83	144
B	74	-	81	145	144	161
C	64	81	-	50	100	80
D	98	145	50	-	83	63
E	83	144	100	83	-	151
F	144	161	80	63	151	-

(a) Use an algorithm to construct a minimum connector between the stations.

(b) Explain how the algorithm can be used in general to provide a lower bound for the solution of the travelling salesman problem, and obtain such a lower bound for the six pumping stations.

(c) Find an upper bound for this problem.

16. The graph below represents a railway network connecting towns A, B, C, D, E, F, and G, with distances between towns shown in kilometres.



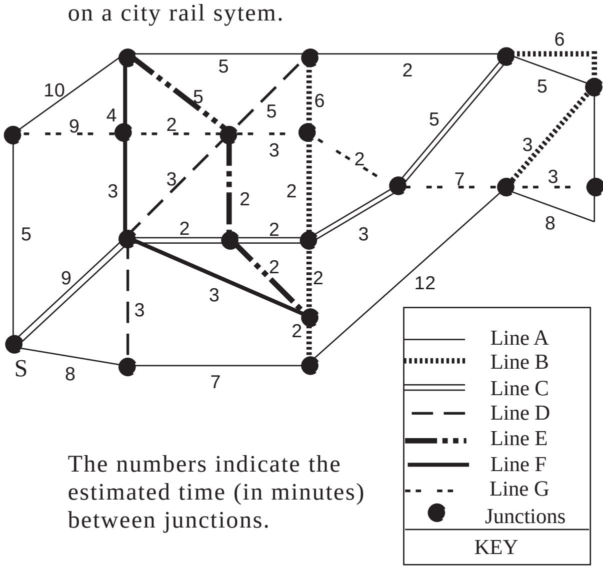
An inspection train is required to begin at B and survey all of the track and return to B. The train needs to travel only in one direction along any section of track in order to carry out the survey of that section.

(a) Explain why it is impossible to complete the survey by travelling along each section of track once and only once.

(b) Find a route of minimum length for the train, stating which section(s) of track must be repeated, and determine the total route length.

(c) Is the solution found in (b) unique? Give an explanation for your answer. (AEB)

17. The network below illustrates the main junctions on a city rail sytem.



The numbers indicate the estimated time (in minutes) between junctions.

(a) Use an appropriate shortest path algorithm to find the path of minimum total time between S and T, assuming that there is no waiting when changing trains at junctions.

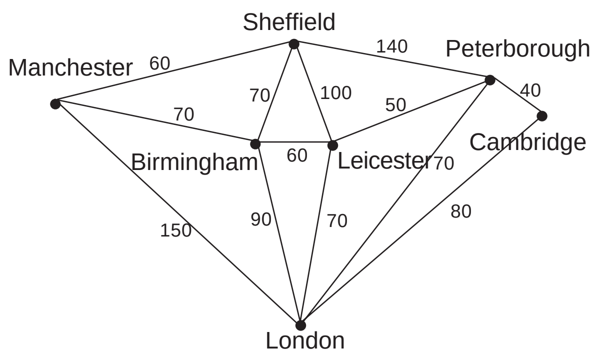
(b) Assuming that an average time for changing and waiting at a junction is 5 minutes, is the solution determined in (a) still valid? If not, what is now the optimal solution?

(Note that, at a junction, you only change trains if you are changing from one line to a different one.) (AEB)

18. The network below illustrates the time (in minutes) of rail journeys between the given cities.

(a) Assuming that any necessary changes of train do not add any time to the journey, find the quickest train route from Manchester to Cambridge.

(b) Assuming that at each city shown a change of train is always necessary, and that each change adds 20 minutes to the journey time, find the quickest train route from Manchester to Cambridge.



(AEB)