15 DIFFERENCE EQUATIONS 2

Objectives

After studying this chapter you should

- be able to obtain the solution of any linear homogeneous second order difference equation;
- be able to apply the method of solution to contextual problems;
- be able to use generating functions to solve non-homogeneous equations.

15.0 Introduction

In order to tackle this chapter you should have studied a substantial part of the previous chapter on first order difference equations. The problems here deal with rather more sophisticated equations, called second order difference equations, which derive from a number of familiar contexts. This is where the rabbits come in.

It is well known that rabbits breed fast. Suppose that you start with one new-born pair of rabbits and every month any pair of rabbits gives birth to a new pair, which itself becomes productive after a period of two months. How many rabbits will there be after n months? The table shows the results for the first few months.

Month	1	2	3	4	5	6
No. of pairs (u_n)	1	1	2	3	5	8

The sequence u_n is a famous one attributed to a 13th century mathematician *Leonardo Fibonacci* (c. 1170-1250). As you can see the next term can be found by adding together the previous two.

The nth term u_n can be written as

$$u_n = u_{n-1} + u_{n-2}$$

and difference equations like this with terms in u_n, u_{n-1} and u_{n-2} are said to be of the **second order** (since the difference between n and $n-2$ is 2).

Activity 1 Fibonacci numbers

The **Fibonacci numbers** have some remarkable properties. If you divide successive terms by the previous term you obtain the sequence,

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \dots = 1, 2, 1.5, 1.\dot{6}, \dots$$

Continue this sequence, say to the 20th term, and find its reciprocal. What do you notice? Can you find an equation with a solution which gives you the limit of this sequence?

Exercise 15A

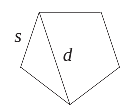
1 If $u_n = 2u_{n-1} + u_{n-2}$ and $u_1 = 2$, $u_2 = 5$, find the values of u_3, u_4, u_5.

2. $u_n = pu_{n-1} + qu_{n-2}$ describes the sequence 1, 2, 8, 20, 68, ... Find p and q.

3. If F_n is a term of the Fibonacci sequence, investigate the value of $F_{n+1}F_{n-1} - F_n{}^2$.

4. What sequences correspond to the difference equation $u_n = u_{n-1} - u_{n-2}$, $n \geq 3$? Choose your own values for u_1 and u_2.

5. Find the ratio of the length of a diagonal to a side of a regular pentagon. What do you notice?



6. Investigate the limit of $\frac{u_n}{u_{n-1}}$ if $u_n = u_{n-1} + 2u_{n-2}$.

7. Show that $F_1 + F_2 + F_3 + ... + F_n = F_{n+2} - 1$, where F_n is a term of the Fibonacci sequence.

15.1 General solutions

When you solved difference equations in the previous chapter, any **general** solution had an unknown constant left in the solution. Usually this was u_1 or u_0.

Example

Solve $u_n = 4u_{n-1} - 3$.

Solution

$$\begin{aligned} u_n &= 4^n u_0 - \frac{3(4^n - 1)}{3} \\ &= 4^n u_0 - (4^n - 1) \\ &= 4^n (u_0 - 1) + 1 \end{aligned}$$

Alternatively you could write

$$u_n = A4^n + 1, \text{ replacing } u_0 - 1 \text{ by A.}$$

This first order equation has one arbitrary constant in its general solution. Knowing the value of u_0 would give you a **particular solution** to the equation.

Say $u_0 = 4$ then

$$4 = A.4^0 + 1 = A + 1.$$

So $A = 3$ and

$$u_n = 3 \times 4^n + 1.$$

In a similar way, **general solutions** to second order equations have two arbitrary constants. Unfortunately, an iterative technique does not work well for these equations, but, as you will see, a guess at the solution being of a similar type to that for first order equations does work.

Suppose

$$\boxed{u_n = pu_{n-1} + qu_{n-2}} \qquad (1)$$

where p, q are constants, $n \geq 2$.

This is a **second order homogeneous linear difference equation** with constant coefficients.

As a solution, try $u_n = Am^n$, where m and A are constants. This choice has been made because $u_n = k^n u_0$ was the solution to the first order equation $u_n = ku_{n-1}$.

Substituting $u_n = Am^n$, $u_{n-1} = Am^{n-1}$ and $u_{n-2} = Am^{n-2}$ into equation (1) gives

$$Am^n = Apm^{n-1} + Aqm^{n-2}$$

$$\Rightarrow \quad Am^{n-2}\left(m^2 - pm - q\right) = 0.$$

If $m = 0$, or $A = 0$, then equation (1) has trivial solutions (i.e. $u_n = 0$). Otherwise, if $m \neq 0$ and $A \neq 0$ then

$$m^2 - pm - q = 0.$$

This is called the **auxiliary equation** of equation (1).

It has the solution

$$m_1 = \frac{p+\sqrt{p^2+4q}}{2} \quad \text{or} \quad m_2 = \frac{p-\sqrt{p^2+4q}}{2}.$$

m_1 and m_2 can be real or complex. The case where $m_1 = m_2$ is special, as you will see later.

Suppose for now that $m_1 \neq m_2$, then it has been shown that both $u_n = Am_1^n$ and $u_n = Bm_2^n$ are solutions of (1), where A and B are constants.

Can you suggest the form of the general solution?

In fact it is easy to show that a **linear** combination of the two solutions is also a solution. This follows since both Am_1^n and Bm_2^n satisfy equation (1) giving

$$Am_1^n = Am_1^{n-1}p + Am_1^{n-2}q$$

and $$Bm_2^n = Bm_2^{n-1}p + Bm_2^{n-2}q$$

$$\Rightarrow \quad Am_1^n + Bm_2^n = p\left(Am_1^{n-1} + Bm_2^{n-1}\right) + q\left(Am_1^{n-2} + Bm_2^{n-2}\right).$$

So $Am_1^n + Bm_2^n$ is also a solution of equation (1), and can in fact be shown to be the general solution. That is **any** solution of (1) will be of this form.

In summary, the general solution of $u_n = pu_{n-1} + qu_{n-2}$ is

$$\boxed{u_n = Am_1^n + Bm_2^n, \quad m_1 \neq m_2}$$

where A, B are arbitrary constants and m_1, m_2 are the solutions of the auxiliary equation $m^2 - pm - q = 0$.

Example

Find the general solution of $u_n = 2u_{n-1} + 8u_{n-2}$.

Solution

The auxiliary equation is $m^2 - 2m - 8 = 0$.

This has solutions $m_1 = 4$ and $m_2 = -2$.

The general solution is therefore

$$u_n = A4^n + B(-2)^n.$$

Example

Solve $u_n + 3u_{n-2} = 0,\ n \geq 3$, given that $u_1 = 1$ and $u_2 = 3$.

Solution

The auxiliary equation is $m^2 + 3 = 0$.

$$\Rightarrow \quad m^2 = -3$$

$$\Rightarrow \quad m_1 = \sqrt{3}\,i \quad \text{and} \quad m_2 = -\sqrt{3}\,i \quad (\text{where } i = \sqrt{-1}\ ).$$

The **general solution** to the equation is therefore

$$u_n = A\left(\sqrt{3}\,i\right)^n + B\left(-\sqrt{3}\,i\right)^n.$$

When $n = 1$, $u_1 = 1$ and since $u_1 = A\left(\sqrt{3}\,i\right)^1 + B\left(-\sqrt{3}\,i\right)^1$

$$\Rightarrow \quad 1 = A\sqrt{3}\,i - B\sqrt{3}\,i$$

$$\frac{1}{\sqrt{3}\,i} = A - B. \qquad (2)$$

When $n = 2$, $u_2 = 3$ and $u_2 = A\left(\sqrt{3}\,i\right)^2 + B\left(-\sqrt{3}\,i\right)^2$

$$\Rightarrow \quad 3 = -A3 - B3$$

$$\Rightarrow \quad -1 = A + B. \qquad (3)$$

Adding (2) and (3) gives

$$2A = -1 + \frac{1}{\sqrt{3}i} = -1 - \frac{i}{\sqrt{3}}$$

$$\Rightarrow \quad A = -\frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right)$$

and

$$B = -\frac{1}{2}\left(1 - \frac{i}{\sqrt{3}}\right).$$

Thus the particular solution to the equation, for $u_1 = 1,\ u_2 = 3$, is given by

$$u_n = -\frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right)\left(\sqrt{3}\,i\right)^n - \frac{1}{2}\left(1 - \frac{i}{\sqrt{3}}\right)\left(-\sqrt{3}\,i\right)^n \qquad (4)$$

Although this solution is given in terms of the complex number $i = \sqrt{-1}$, it is in fact always a real number.

Activity 2

Show that equation (4) gives $u_1 = 1$ and $u_2 = 3$. Also use this equation to evaluate u_3 and u_4, and check these answers directly from the original difference equation, $u_n + 3u_{n-2} = 0$.

Exercise 15B

1. Find the general solutions to

   (a) $u_n = u_{n-1} + 6u_{n-2}$

   (b) $u_n = 4u_{n-1} + u_{n-2}$

   (c) $u_n - u_{n-1} - 2u_{n-2} = 0$.

2. Find the general solution of the difference equation associated with the Fibonacci sequence. Use $u_0 = 1$, $u_1 = 1$, to find the particular solution.

3. Solve $u_n + 4u_{n-2} = 0$, $n \geq 3$, if $u_1 = 2$, $u_2 = -4$.

4. Solve $u_n - 6u_{n-1} + 8u_{n-2} = 0$, $n \geq 3$, given $u_1 = 10$, $u_2 = 28$. Evaluate u_6.

5. Find the nth term of the sequence

   –3, 21, 3, 129, 147 ...

15.2 Equations with equal roots

When $m_1 = m_2$, the solution in Section 15.1 would imply that

$$\begin{aligned} u_n &= Am_1^n + Bm_1^n \\ &= m_1^n (A + B) \\ &= m_1^n C, \quad \text{where } C = A + B. \end{aligned}$$

In this case there is really only one constant, compared with the two expected. Trials show that another possibility for a solution to

$$u_n = pu_{n-1} + qu_{n-2} \qquad (1)$$

is $u_n = Dnm_1^n$, and as you will see below, this solution, combined with one of the form Cm_1^n, gives a general solution to the equation when $m_1 = m_2$.

If $u_n = Dnm_1^n$ then

$$u_{n-1} = D(n-1)\,m_1^{n-1}$$

and

$$u_{n-2} = D(n-2)\,m_1^{n-2}.$$

If $u_n = Dnm_1^n$ is a solution of (1), then $u_n - p\,u_{n-1} - q\,u_{n-2}$ should equal zero.

$$u_n - pu_{n-1} - qu_{n-2}$$

$$= Dnm_1^n - pD(n-1)m_1^{n-1} - qD(n-2)m_1^{n-2}$$

$$= Dm_1^{n-2}\left[nm_1^2 - (n-1)pm_1 - (n-2)q\right]$$

$$= Dm_1^{n-2}\left[n\left(m_1^2 - pm_1 - q\right) + pm_1 + 2q\right]$$

$$= Dm_1^{n-2}\left(pm_1 + 2q\right)$$

because $m_1^2 - pm_1 - q = 0$.

Now, the auxiliary equation has equal roots, which means that

$$p^2 + 4q = 0 \quad \text{and} \quad m_1 = \frac{p}{2}.$$

Therefore $u_n - pu_{n-1} - qu_{n-2} = Dm_1^{n-2}\left(p \times \frac{p}{2} + 2q\right)$

$$= 2Dm_1^{n-2}\left(p^2 + 4q\right)$$

$$= 0, \text{ since } p^2 + 4q = 0.$$

So $u_n = Dnm_1^n$ is a solution and therefore $Dnm_1^n + Cm_1^n$ will be also. This can be shown by using the same technique as for the case when $m_1 \neq m_2$.

In summary, when $p^2 + 4q = 0$ the general solution of $u_n = pu_{n-1} + qu_{n-2}$ is

$$\boxed{u_n = Cm_1^n + Dnm_1^n}$$

where C and D are arbitrary constants.

Example

Solve $u_n + 4u_{n-1} + 4u_{n-2} = 0$, $n \geq 3$, if $u_1 = -2$ and $u_2 = 12$. Evaluate u_5.

Solution

The auxiliary equation is

$$m^2 + 4m + 4 = 0$$

$$\Rightarrow \quad (m+2)^2 = 0$$

$$\Rightarrow \quad m_1 = m_2 = -2.$$

Therefore the general solution is

$$u_n = Dn(-2)^n + C(-2)^n$$

or $$u_n = (-2)^n(C + Dn).$$

If $u_1 = -2, \quad -2(C + D) = -2 \quad \Rightarrow \quad C + D = 1.$

Also, as $u_2 = 12, \quad 4(C + 2D) = 12 \quad \Rightarrow \quad C + 2D = 3.$

These simultaneous equations can be solved to give $C = -1$ and $D = 2$.

Thus $$u_n = (-2)^n(2n - 1)$$

and $$u_5 = (-2)^5(10 - 1) = -32 \times 9 = -288.$$

Activity 3

Suppose that a pair of mice can produce two pairs of offspring every month and that mice can reproduce two months after birth. A breeder begins with a pair of new-born mice. Investigate the number of mice he can expect to have in successive months. You will have to assume no mice die and pairs are always one female and one male!

If a breeder begins with ten pairs of mice, how many can he expect to have bred in a year?

Exercise 15C

1. Find the general solutions of

   (a) $u_n - 4u_{n-1} + 4u_{n-2} = 0$

   (b) $u_n = 2u_{n-1} - u_{n-2}$.

2. Find the particular solution of

   $u_n - 6u_{n-1} + 9u_{n-2} = 0$, $n \geq 3$, when $u_1 = 9$, $u_2 = 36$.

3. If $u_1 = 0, u_2 = -4$, solve $u_{n+2} + u_n = 0$, $n \geq 1$, giving u_n in terms of i.

4. Find the particular solution of

   $u_{n+2} + 2u_{n+1} + u_n = 0$, $n \geq 1$, when $u_1 = -1$, $u_2 = -2$.

5. Find the solutions of these difference equations.

   (a) $u_n - 2u_{n-1} - 15u_{n-2} = 0$, $n \geq 3$, given $u_1 = 1$ and $u_2 = 77$.

   (b) $u_n = 3u_{n-2}$, $n \geq 3$, given $u_1 = 0$, $u_2 = 3$.

   (c) $u_n - 6u_{n-1} + 9u_{n-2} = 0$, $n \geq 3$, given $u_1 = 9$, $u_2 = 45$.

6. Form and solve the difference equation defined by the sequence in which the *n*th term is formed by adding the previous two terms and then doubling the result, and in which the first two terms are both one.

15.3 A model of the economy

In good times, increased national income will promote increased spending and investment.

If you assume that government expenditure is constant (G) then the remaining spending can be assumed to be composed of investment (I) and private spending on consumables (P). So you can model the national income (N) by the equation

$$N_t = I_t + P_t + G, \text{ where } t \text{ is the year number} \qquad (1)$$

If income increases from year $t-1$ to year t, then you would assume that private spending will increase in year t proportionately. So you can write :

$$P_t = AN_{t-1}, \quad \text{where } A \text{ is a constant.}$$

Also, extra private spending should promote additional investment. So you can write :

$$I_t = B(P_t - P_{t+1}), \quad \text{where } B \text{ is a constant.}$$

Substituting for P_t and I_t in (1) gives

$$N_t = AN_{t-1} + B(P_t - P_{t-1}) + G$$

$$= AN_{t-1} + B(AN_{t-1} - AN_{t-2}) + G$$

$$N_t = A(B+1)N_{t-1} - ABN_{t-2} + G. \qquad (2)$$

So far you have not met equations of this type in this chapter. It is a second order difference equation, but it has an extra constant G.

Before you try the activity below, discuss the effects you think the values of A and B will have on the value of N as t increases.

Activity 4

Take, as an example, an economy in which for year 1, $N_1 = 2$ and for year 2, $N_2 = 4$. Suppose that $G = 1$. By using the difference equation (2) above, investigate the change in the size of N over a number of years for different values of A and B.

At this stage you should not attempt an algebraic solution!

Equation (2) is an example of a **non-homogeneous difference equation**. **Homogeneous** second order equations have the form

$$\boxed{u_n + au_{n-1} + bu_{n-2} = 0}$$

There are no other terms, unlike equation (2) which has an additional constant G.

15.4 Non-homogeneous equations

You have seen how to solve homogeneous second order difference equations; i.e. ones of the form given below but where the right-hand side is zero. Turning to non-homogeneous equations of the form

$$\boxed{u_n + au_{n-1} + bu_{n-2} = f(n)}$$

where f is a function of n, consider as a first example the equation

$$6u_n - 5u_{n-1} + u_{n-2} = n, \quad (n \geq 3) \qquad (1)$$

Activity 5

Use a computer or calculator to investigate the sequence u_n defined by

$$6u_n - 5u_{n-1} + u_{n-2} = n, \quad (n \geq 3)$$

for different starting values. Start, for example, with $u_1 = 1$, $u_2 = 2$ and then vary either or both of u_1 and u_2. How does the sequence behave when n is large?

From the previous activity, you may have had a feel for the behaviour or structure of the solution. Although its proof is beyond the scope of this text, the result can be expressed as

$$u_n = \begin{pmatrix} \text{general solution of} \\ \text{associated homogeneous} \\ \text{equation} \end{pmatrix} + \begin{pmatrix} \text{one particular} \\ \text{solution of the} \\ \text{full equation} \end{pmatrix}$$

That is, to solve

$$6u_n - 5u_{n-1} + u_{n-2} = n, \quad (n \geq 2)$$

you first find the general solution of the associated homogeneous equation

$$6u_n - 5u_{n-1} + u_{n-2} = 0 \qquad (2)$$

and, to this, add one particular solution of the full equation.

You have already seen in Section 15.2 how to solve equation (2). The auxiliary equation is

$$6m^2 - 5m + 1 = 0$$

$$\Rightarrow \quad (3m-1)(2m-1) = 0$$

$$\Rightarrow \quad m = \tfrac{1}{3} \text{ or } \tfrac{1}{2}.$$

So the general solution of (2) is given by

$$u_n = A\left(\tfrac{1}{3}\right)^n + B\left(\tfrac{1}{2}\right)^n, \qquad (3)$$

where A and B are constants.

The next stage is to find one particular solution of the full equation (1).

Can you think what type of solution will satisfy the full equation?

In fact, once you have gained experience in solving equations of this type, you will recognise that u_n will be of the form

$$u_n = a + bn$$

(which is a generalisation of the function on the right-hand side, namely n).

So if $\quad u_n = a + bn$

$$\Rightarrow \quad u_{n-1} = a + b(n-1)$$

$$\Rightarrow \quad u_{n-2} = a + b(n-2)$$

and to satisfy (1), we need

$$6(a+bn) - 5\big(a + b(n-1)\big) + a + b(n-2) = n$$

$$6a + 6bn - 5a - 5bn + 5b + a + bn - 2b = n$$

$$2a + 3b + n(2b) = n.$$

Each side of this equation is a polynomial of degree 1 in n.

How can both sides be equal?

To ensure that it is satisfied for all values of n, equate cooefficients on each side of the equation.

constant term $\Rightarrow$ $2a + 3b = 0$

n term $\Rightarrow$ $2b = 1$.

So $b = \frac{1}{2}$ and $a = -\frac{3}{4}$, and you have shown that one particular solution is given by

$$u_n = -\tfrac{3}{4} + \tfrac{1}{2}n. \qquad (4)$$

To complete the general solution, add (4) to (3) to give

$$u_n = A\left(\tfrac{1}{3}\right)^n + B\left(\tfrac{1}{2}\right)^n - \tfrac{3}{4} + \tfrac{1}{2}n.$$

Activity 6

Find the solution to equation (1) which satisfies $u_1 = 1,\ u_2 = 2$.

The main difficulty of this method is that you have to 'guess' the form of the particular solution. The table below gives the usual form of the solution for various functions $f(n)$.

$f(n)$	**Form of particular solutions**
constant	a
n	$a + bn$
n^2	$a + bn + cn^2$
k^n	ak^n (or ank^n in special cases)

The next three examples illustrate the use of this table.

Example

Find the general solution of $6u_n - 5u_{n-1} + u_{n-2} = 2$.

Solution

From earlier work, the form of the general solution is

$$u_n = A\left(\tfrac{1}{3}\right)^n + B\left(\tfrac{1}{2}\right)^n + \begin{pmatrix}\text{one particular}\\ \text{solution}\end{pmatrix}$$

For the particular solution, try

$$u_n = a \quad \Rightarrow \quad u_{n-1} = a \quad \text{and} \quad u_{n-2} = a$$

which on subtituting in the equation gives

$$6a - 5a + a = 2 \;\Rightarrow\; a = 1.$$

Hence $u_n = 1$ is a particular solution and the general solution is given by

$$u_n \;=\; A\left(\tfrac{1}{3}\right)^n + B\left(\tfrac{1}{2}\right)^n + 1.$$

Example

Find the general solution of $6u_n - 5u_{n-1} + u_{n-2} = 2^n$.

Solution

For the particular solution try $u_n = a2^n$, so that $u_{n-1} = a2^{n-1}$, and substituting in the equation gives

$$6a2^n - 5a2^{n-1} + a2^{n-2} \;=\; 2^n$$

$$2^{n-2}(6a \times 4 - 5a \times 2 + a) \;=\; 2^n$$

$$24a - 10a + a \;=\; 4$$

$$\Rightarrow \qquad a \;=\; \tfrac{4}{15}$$

and so the general solution is given by

$$u_n \;=\; A\left(\tfrac{1}{3}\right)^n + B\left(\tfrac{1}{2}\right)^n + \left(\tfrac{4}{15}\right)2^n.$$

In the next example, the equation is

$$6u_n - 5u_{n-1} + u_{n-2} \;=\; \left(\tfrac{1}{2}\right)^n.$$

Can you see why the usual trial for a particular solution, namely $u_n = a\left(\tfrac{1}{2}\right)^n$ will not work?

Example

Find the general solution of $6u_n - 5u_{n-1} + u_{n-2} = \left(\tfrac{1}{2}\right)^n$.

Solution

If you try $u_n = a\left(\tfrac{1}{2}\right)^n$ for a particular solution, you will not be able to find a value for the constant a to give a solution. This is because the term $B\left(\tfrac{1}{2}\right)^n$ is already in the solution of the associated homogeneous equation. In this special case, try

$$u_n = an\left(\tfrac{1}{2}\right)^n$$

so that $u_{n-1} = a(n-1)\left(\tfrac{1}{2}\right)^{n-1}$

and $u_{n-2} = a(n-2)\left(\tfrac{1}{2}\right)^{n-2}$.

Substituting in the equation gives

$$6an\left(\tfrac{1}{2}\right)^n - 5a(n-1)\left(\tfrac{1}{2}\right)^{n-1} + a(n-2)\left(\tfrac{1}{2}\right)^{n-2} = \left(\tfrac{1}{2}\right)^n$$

$$\left(\tfrac{1}{2}\right)^{n-2}\left(6an\left(\tfrac{1}{2}\right)^2 - 5a(n-1)\tfrac{1}{2} + a(n-2)\right) = \left(\tfrac{1}{2}\right)^n$$

$$\tfrac{3}{2}an - \tfrac{5}{2}an + \tfrac{5}{2}a + an - 2a = \tfrac{1}{4}$$

$$\tfrac{1}{2}a = \tfrac{1}{4} \quad \text{(the } n \text{ terms cancel out).}$$

Hence $a = \tfrac{1}{2}$, and the particular solution is

$$u_n = \tfrac{1}{2}n\left(\tfrac{1}{2}\right)^n = n\left(\tfrac{1}{2}\right)^{n+1}.$$

The general solution is given by

$$u_n = A\left(\tfrac{1}{3}\right)^n + B\left(\tfrac{1}{2}\right)^n + n\left(\tfrac{1}{2}\right)^{n+1}.$$

The examples above illustrate that, although the algebra can become quite complex, the real problem lies in the intelligent choice of the form of the solution. Note that if the usual form does not work, then the degree of the polynomial being tried should be increased by one. The next two sections will illustrate a more general method, not dependent on inspired guesswork!

Exercise 15D

1. Find the general solution of the difference equation $u_n - 5u_{n-1} + 6u_{n-2} = f(n)$ when

   (a) $f(n) = 2$ (b) $f(n) = n$

   (c) $f(n) = 1 + n^2$ (d) $f(n) = 5^n$

   (e) $f(n) = 2^n$.

2. Find the complete solution of

   $$u_n - 7u_{n-1} + 12u_{n-2} = 2^n$$

   when $u_1 = 1$ and $u_2 = 1$.

3. Find the general solution of

   $$u_n + 3u_{n-1} - 10u_{n-2} = 2^n$$

   and determine the solution which satisfies $u_1 = 2$, $u_2 = 1$.

15.5 Generating functions

This section will introduce a new way of solving difference equations by first applying the method to homogeneous equations.

A different way of looking at a sequence u_0, u_1, u_2, u_3 is as the coefficients of a power series

$$G(x) = u_0 + u_1 x + u_2 x^2 + \ldots$$

Notice that the sequence and series begin with u_0 rather than u_1. This makes the power of x and the suffix of u the same, and will help in the long run.

$G(x)$ is called the **generating function** for the sequence $u_0, u_1, u_2, \ldots$ This function can be utilised to solve difference equations. Here is an example of a type you have already met, to see how the method works.

Example

Solve $u_n = 3u_{n-1} - 2u_{n-2} = 0,\ n \geq 2$, given $u_0 = 2,\ u_1 = 3$.

Solution

Let
$$G(x) = u_0 + u_1 x + u_2 x^2 + \ldots$$
$$= 2 + 3x + u_2 x^2 + \ldots \qquad (1)$$

Now from the original difference equation

$$u_2 = 3u_1 - 2u_0$$
$$u_3 = 3u_2 - 2u_1$$
$$u_4 = 3u_3 - 2u_2, \quad \text{etc.}$$

Substituting for $u_2, u_3, u_4, \ldots$ into equation (1) gives

$$G(x) = 2 + 3x + (3u_1 - 2u_0)x^2 + (3u_2 - 2u_1)x^3 + \ldots$$

$$= 2 + 3x + \left(3u_1 x^2 + 3u_2 x^3 + 3u_3 x^4 + \ldots\right) - \left(2u_0 x^2 + 2u_1 x^3 + 2u_2 x^4 + \ldots\right)$$

$$= 2 + 3x + 3x\left(u_1 x + u_2 x^2 + u_3 x^3 + \ldots\right) - 2x^2\left(u_0 + u_1 x + u_2 x^2 + \ldots\right)$$

$$= 2 + 3x + 3x\left(G(x) - u_0\right) - 2x^2 G(x)$$

$$= 2 + 3x + 3x\left(G(x) - 2\right) - 2x^2 G(x)$$

$$G(x) = 2 - 3x + 3x\,G(x) - 2x^2 G(x).$$

Rearranging so that $G(x)$ is the subject gives

$$G(x) = \frac{2 - 3x}{1 - 3x + 2x^2}.$$

Note that $1 - 3x + 2x^2$ is similar to the auxiliary equation you met previously but **not** the same.

Factorising the denominator gives

$$G(x) = \frac{2 - 3x}{(1 - 2x)(1 - x)}. \qquad (2)$$

You now use partial fractions in order to write $G(x)$ as a sum of two fractions. There are a number of ways of doing this which you should have met in your pure mathematics core studies.

So $\qquad G(x) = \dfrac{1}{1 - 2x} + \dfrac{1}{1 - x}.$

Now both parts of $G(x)$ can be expanded using the binomial theorem

$$(1 - 2x)^{-1} = \left(1 + 2x + (2x)^2 + (2x)^3 + \ldots\right)$$

and $\qquad (1 - x)^{-1} = \left(1 + x + x^2 + x^3 + \ldots\right)$.

This gives

$$\ldots x + (2x)^2 + \ldots\Big) + \Big(1 + x + x^2 + \ldots\Big)$$

$$= (1 + 1) + (2x + x) + \left(2^2 x^2 + x^2\right)$$

$$+ \left(2^3 x^3 + x^x\right) + \ldots$$

$$= 2 + 3x + \left(2^2 + 1\right)x^2 + \left(2^3 + 1\right)x^3 + \ldots$$

As you can see, the nth term of $G(x)$ is $\left(2^n + 1\right)x^n$ and the coefficient of x^n is simply u_n - the solution to the difference equation.

So $\qquad u_n = 2^n + 1.$

*Exercise 15E

1. Find the generating function associated with these difference equations and sequences.

   (a) $u_n = 2u_{n-1} + 8u_{n-2}$, given $u_0 = 0$, $u_1 = 1$, $n \ge 2$.

   (b) $u_n + u_{n-1} - 3u_{n-2} = 0$, given $u_0 = 2$, $u_1 = 5$, $n \ge 2$.

   (c) $u_n = 4u_{n-2}$, given $u_0 = 1$, $u_1 = 3$, $n \ge 2$.

   (d) 1, 2, 4, 8, 16, ...

2. Write these expressions as partial fractions:

   (a) $\dfrac{3x-5}{(x-3)(x+1)}$ (b) $\dfrac{1}{(2x-5)(x-2)}$

   (c) $\dfrac{x+21}{x^2-9}$

3. Write these expressions as power series in x, giving the nth term of each series :

   (a) $\dfrac{1}{1-x}$ (b) $\dfrac{1}{1-2x}$ (c) $\dfrac{1}{1+3x}$

   (d) $\dfrac{1}{(1-x)^2}$ (e) $\dfrac{3}{(1+2x)^2}$

4. Solve these difference equations by using generating functions :

   (a) $u_n - 3u_{n-1} + 4u_{n-2} = 0$, given $u_0 = 0$, $u_1 = 20$, $n \ge 2$.

   (b) $u_n = 4u_{n-1}$, given $u_0 = 3$, $n \ge 1$.

5. Find the generating function of the Fibonacci sequence.

6. Find the particular solution of the difference equation $u_{n+2} = 9u_n$, given $u_0 = 5$, $u_1 = -3$, $n \ge 0$.

15.6 Extending the method

This final section shows how to solve non-homogeneous equations of the form :

$$\boxed{u_n + au_{n-1} + bu_{n-2} = f(n)} \quad (a,\ b \text{ constants}) \qquad (1)$$

using the generating function method. The techniques which follow will also work for first order equations (where $b = 0$).

Example

Solve $u_n + u_{n-1} - 6u_{n-2} = n$, $n \ge 2$, given $u_0 = 0$, $u_1 = 2$.

Solution

Let the generating function for the equation be

$$G(x) = u_0 + u_1 x + u_2 x^2 + \ldots.$$

Now work out $\left(1 + x - 6x^2\right)G(x)$.

The term $\left(1 + x - 6x^2\right)$ comes from the coefficients of u_n, u_{n-1} and u_{n-2} in the equation.

Now $\left(1+x-6x^2\right)G(x) = \left(1+x-6x^2\right)\left(u_0+u_1x+u_2x^2+\ldots\right)$

$$= u_0+\left(u_1+u_2\right)x+\left(u_2+u_1-6u_0\right)x^2+\ldots$$

$$= 0+2x+2x^2+3x^3+4x^4+\ldots \qquad (2)$$

In the last step values have been substituted for $u_n+u_{n-1}-6u_{n-2}$.
For example $u_3+u_2-6u_1=3$.

The process now depends on your being able to sum the right-hand side of equation (2). The difficulty depends on the complexity of $f(n)$.

You should recognise (from Exercise 15E , Question 3) that

$$1+2x+3x^2+4x^3+\ldots = \frac{1}{(1-x)^2}.$$

So $\left(1+x-6x^2\right)G(x) = x\left(2+2x+3x^2+4x^3+\ldots\right)$

$$= 2x+x\left(2x+3x^2+4x^3+\ldots\right)$$

$$= 2x+x\left(\frac{1}{(1-x)^2}-1\right)$$

$$= 2x+\frac{x}{(1-x)^2}-x$$

$$= x+\frac{x}{(1-x)^2}.$$

$$\Rightarrow (1+3x)(1-2x)G(x) = \frac{x(1-x)^2+x}{(1-x)^2}$$

$$= \frac{x^3-2x^2+2x}{(1-x)^2}$$

$$\Rightarrow \qquad G(x) = \frac{x^3-2x^2+2x}{(1+3x)(1-2x)(1-x)^2}.$$

This result has to be reduced to partial fractions.

Let $G(x) \equiv \frac{A}{1+3x} + \frac{B}{1-2x} + \frac{C}{(1-x)^2} + \frac{D}{1-x}$

$$\equiv \frac{x^3-2x^2+2x}{(1-3x)(1-2x)(1-x)^2},$$

then solving in the usual way gives $A=-\frac{5}{16}$, $B=1$, $C=-\frac{1}{4}$, $D=-\frac{7}{16}$.

Thus $G(x) = \frac{-5}{16(1+3x)} + \frac{1}{(1-2x)} - \frac{1}{4(1-x)^2} - \frac{7}{16(1-x)}$

$$= -\tfrac{5}{16}\left(1-3x+(-3x)^2+\ldots\right) + \left(1+2x+(2x)^2+\ldots\right)$$

$$-\tfrac{1}{4}\left(1+2x+3x^2+\ldots\right) - \tfrac{7}{16}\left(1+x+x^2+\ldots\right).$$

Picking out the coefficient of the nth term in each bracket gives

$$u_n = -\tfrac{5}{16}(-3)^n + 2^n - \tfrac{1}{4}(n+1) - \tfrac{7}{16}$$

$$= \tfrac{1}{16}\left[-5(-3)^n + 16\times 2^n - 4(n+1) - 7\right].$$

As you can see, the form of u_n is still $Am_1^n + Bm_2^n$, but with the addition of a term of the form $Cn+D$. This additional term is particular to the function $f(n)$, which was n in this case, and to the values of u_0 and u_1.

Exercise 15F

1. Write the following as partial fractions.

   (a) $\frac{1}{(1-x)(1-2x)}$ (b) $\frac{2x-3}{(2-x)(1+x)}$

   (c) $\frac{x^2+2}{(x+1)(1-2x)^2}$

2. Sum these series. Each is the result of expanding the expression of the form $(a+bx)^n$ using the Binomial Theorem (commonly $n=-1$ or -2).

   (a) $1+x+x^2+x^3+\ldots$

   (b) $1+2x+3x^2+4x^3+5x^4+\ldots$

   (c) $-2-3x-4x^2-5x^3-6x^4-..$

   (d) $x^2+2x^3+3x^4+4x^5+\ldots$

   (e) $1+2x+(2x)^2+(2x)^3+(2x)^4+\ldots$

   (f) $(5x)^2+(5x)^3+(5x)^4+\ldots.$

3. Expand as power series :

   (a) $(1-3x)^{-1}$ (b) $\frac{1}{(2-x)^2}$ (c) $\frac{x}{x-1}$.

   Give the nth term in each case.

4. Using the generating function method, solve

   (a) $u_n - 2u_{n-1} - 8u_{n-2} = 8$, $n \geq 2$, given $u_0 = 0$, $u_1 = 2$.

   (b) $u_n - 2u_{n-1} = 3^n$, $n \geq 1$, given $u_0 = 1$.

   (c) $u_n - u_{n-1} - 2u_{n-2} = n^2$, $n \geq 2$, given $u_0 = 0$, $u_1 = 1$.

15.7 Miscellaneous Exercises

1. Solve $u_n - 4u_{n-2} = 0,\ n \geq 3,$ when $u_1 = 2$ and $u_2 = 20$.

2. Find the general solution of $u_n - 4u_{n-1} + 4u_{n-2} = 0$.

3. The life of a bee is quite amazing. There are basically three types of bee :

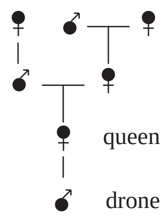
   queen a fertile female

   worker an infertile female

   drone a fertile male.

   Eggs are either fertilised, resulting in queens and workers or unfertilised, resulting in drones.

   Trace back the ancestors of a drone. Find the numbers of ancestors back to the nth generation. The generation tree has been started for you below :



4. Find the nth term of these sequences :

   (a) 2, 5, 11, 23, ...

   (b) 2, 5, 12, 27, 58, ...

   (c) 1, 2, 6, 16, 44, ...

5. Solve the difference equation

   $$u_n - u_{n-1} - 12u_{n-2} = 2^n,\ n \geq 2,$$

   if $u_0 = 0$, and $u_1 = 1$.

6. Write down the general solution of the model of the economy in Activity 4 when $A = \frac{2}{3}$, $B = 4$ and $N_1 = 1$, $N_2 = 2$.

7. A difference equation of the form

   $$u_n + au_{n-1} + bu_{n-2} = k$$

   defines a sequence with its first five terms as 0, 2, 5, 5, 14. Find the nth term.

8. Find the smallest value of n for which u_n exceeds one million if $u_n = 10 + 3u_{n-1}$, $n \geq 1$, given $u_0 = 0$.

9. Find the solution of the difference equation

   $$u_n = u_{n-2} + n,\ n \geq 2,$$

   given $u_1 = u_0 = 1$.

10. (a) Solve, by iteration, the recurrence relation

   $$u_n = 4 - 3u_{n-1},$$

   (i) subject to the initial condition $u_1 = 10$;

   (ii) subject to the initial condition $u_1 = 1$.

   (b) A ternary sequence is a sequence of numbers, each of which is 0, 1 or 2. (For example, 1002 and 1111 are 4-digit ternary sequences.) Let u_n be the number of n-digit ternary sequences which do not contain two consecutive 0s. By considering the number of such sequences which begin with 0, and the number which begin with 1 or 2, find a second-order linear recurrence relation for u_n, and write down appropriate initial conditions.

   (c) Using the method of generating functions, solve the recurrence relation $u_n - 5u_{n-1} + 6u_{n-2} = 0$, subject to the initial conditions $u_0 = 1$, $u_1 = 3$.

11. Write down the general solution of the difference equation

   $$u_{n+1} = 3u_n,\ n \geq 1.$$

   Hence solve the difference equation

   $$u_{n+1} = 3u_n + 5, \text{ given that } u_1 = 6.$$

12. In a new colony of geese there are 10 pairs of birds, none of which produce eggs in their first year. In each subsequent year, pairs of birds which are in their second or later year have, on average, 4 eggs (2 male and 2 female)). Assuming no deaths, show that the recurrence relation which describes the geese population is

   $$u_{r+1} = u_r + 2u_{r-1},\ u_1 = 10 \text{ and } u_2 = 10,$$

   where u_r represents the geese population (in pairs) at the beginning of the rth year.

13. A pair of hares requires a maturation period of one month before they can produce offspring. Each pair of mature hares present at the end of one month produces two new pairs by the end of the next month. If u_n denotes the number of pairs alive at the end of the nth month and no hares die, show that u_n satisfies the recurrence relation $u_n = u_{n-1} + 2u_{n-2}$.

   Solve this recurrence relation subject to the initial conditions $u_0 = 6$ and $u_1 = 9$.

   Also find the solution for u_n if the initial conditions $u_0 = 4$ and $u_1 = 8$. In this case, how many months will it take for the hare population to be greater than 1000?

14. Find the general solution of the recurrence relation $u_n + 3u_{n-1} - 4u_{n-2} = 0$.

15.(a) Solve the recurrence relation $u_n = (n-1)u_{n-1}$, subject to the initial condition $u_1 = 3$.

(b) Find the general solution of the recurrence relation $u_n + 4u_{n-1} + 4u_{n-2} = 0$.

16. In an experiment the pressure of gas in a container is measured each second and the pressure (in standard units) in n seconds is denoted by p_n. The measurements satisfy the recurrence relation

$p_0 = 6$, $p_1 = 3$, and $p_{n+2} = \frac{1}{2}\left(p_{n+1} + p_n\right)$ $(n \geq 0)$. Find an explicit formula for p_n in terms of n, and state the value to which the pressure settles down in the long term.

17. The growth in number of neutrons in a nuclear reaction is modelled by the recurrence relation $u_{n+1} = 6u_n - 8u_{n-1}$, with initial values $u_1 = 2$, $u_2 = 5$, where u_n is the number at the beginning of the time interval n $(n = 1, 2, \ldots)$. Find the solution for u_n and hence, or otherwise, determine the value of n for which the number reaches 10 000.

(AEB)

18. A population subject to natural growth and harvesting is modelled by the recurrence relation $u_{n+1} = (1+\alpha)u_n - k2^n$. Here u_n denotes the population size at time n and α and k are positive numbers. If $u_0 = a$, find the solution for u_n in terms of n, α, a and k.

(a) With no harvesting $(k = 0)$, $a = 100$ and $\alpha = 0.2$, determine the smallest value of n for which $u_n \geq 200$.

(b) With $k = 1$, $a = 200$ and $\alpha = 0.2$,

(i) Show that $u_n = (201.25)(1.2)^n - (1.25)2^n$ $(n = 0, 1, 2, \ldots)$

(ii) What is the long term future of the population?

(iii) Determine the value of n which gives the greatest population.

(AEB)