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Chapter 10   Logic

10 LOGIC

Objectives

After studying this chapter you should

- understand the nature of propositional logic;
- understand the connectives NOT, OR, AND;
- understand implication and equivalence;
- be able to use truth tables;
- be able to identify tautology and contradiction;
- be able to test the validity of an argument.

10.0 Introduction

It may seem unusual for philosophical ideas of logic based on intuition to be represented mathematically, however, the mathematics that has developed to describe logic has, in recent years, been crucial in the design of computer circuits and in automation.

Charles L Dodgson (1832 -1898) who under the pseudonym *Lewis Carroll* wrote 'Alice in Wonderland', was an Oxford mathematician who wrote about logic. One example of his logic problems concerns Mrs Bond's ducks.

Activity 1   Do ducks wear collars?

The following lines are taken from Lewis Carroll's book 'Symbolic Logic' first published in 1897.

> "All ducks in this village, that are branded 'B' belong to Mrs Bond;
>
> Ducks in this village never wear lace collars, unless they are branded 'B';
>
> Mrs Bond has no grey ducks in this village."

Is the conclusion 'no grey ducks in this village wear lace collars' valid?

10.1 The nature of logic

The Greek philosopher *Aristotle* (384-322 BC) is considered to be the first to have studied logic in that he formed a way of representing **logical propositions** leading to a conclusion. Aristotle's theory of **syllogisms** provides a way of analysing **propositions** given in the form of statements.

For example, here are some propositions.

All apples are fruits.

No toothache is pleasant.

Some children like chocolate.

Some cheese is not pasteurised.

In each of these statements a **subject**, S (e.g. apples) is linked to a **predicate**, P (e.g. fruits). The **quantity** of each subject is indicated by the word 'all', 'no' or 'some'.

Statements can also be described as **universal** ('all' or 'no') or **particular** ('some') and **affirmative** or **negative**.

The four statements above can be described in this way:

all S is P   universal affirmative

no S is P   universal negative

some S is P   particular affirmative

some S is not P   particular negative .

Aristotle described an argument by linking together three statements; two statements, called **premises,** lead to a third statement which is the **conclusion** based on the premises. This way of representing an **argument** is called a **syllogism.**

Example

If fruits are tasty

and apples are fruits

then apples are tasty.

In this example of a syllogism the conclusion of the argument has apples as subject (S) and tasty as predicate (P). The first premise includes P and the second premise includes S and both the premises include 'fruit', which is known as the middle term (M).

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Chapter 10   Logic

The syllogism can therefore be described as:

If   M   is   P

and   S   is   M

then   S   is   P.

Activity 2   Finding the figure

By removing all the words the last example can be described as

M P

S M
___
S P

On the assumption that

- the conclusion of the argument must be SP,
- the first premise must contain P,
- the second premise must contain S,
- both the first and the second premise must contain M,

find the three other arrangements.

Together these are known as the **four figures** of the syllogism.

Not all syllogisms are valid

Each of the four figures can be universal or particular, affirmative or negative, but not all these arrangements give valid arguments.

Example

Is this valid?

No M is P

All S is M
___
Some S is P.

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Chapter 10 Logic

Solution

An example of this syllogism might be:

No animal with 4 legs is a bird

All cats are animals with 4 legs
———————————————
Some cats are birds.

Obviously this arrangement is an invalid argument!

Activity 3   Valid or invalid?

Decide if these syllogisms are valid.

(a)	Some M is P	(b)	All P is M
	All S is M		No S is M
	All S is P		No S is P

(c)	No M is P	(d)	Some P is M
	All M is S		All M is S
	Some S is P		Some S is P

10.2 Combining propositions

Modern logic is often called **propositional logic**; the word
'proposition' is defined as a statement that is either **true** or **false**.
So far, a variety of propositions have been considered, such as
premises and conclusions to an argument.

For example, consider the statement

'the water is deep'.

It is not possible to say if this is true or false unless the word
'deep' is defined and, without a precise definition, this cannot be
called a proposition.

Example

- **p** stands for the proposition 'January has 31 days',
    which is true.

- **q** stands for the proposition '$4 + 7 = 10$',   which is false.

- 'What a hot day' is not a proposition because it is not in
    subject-predicate form; also the word 'hot' is not defined.

Negation   NOT   ~

Each proposition has a corresponding negation and, if the proposition is denoted by **p**, the negation of the proposition is denoted by ~**p**, read as 'not **p**'.

Example

If **p** is the proposition 'the table is made of pine',

then   ~**p** is the proposition 'the table is not made of pine'.

If **q** is the proposition 'the sack is empty', then ~ q̰ is the statement 'the sack is not empty'. It is not correct to assume that the negation is 'the sack is full', since the statement 'the sack is not empty' could mean 'the sack is only partly full'.

Connectives

Simple propositions such as

> 'Elgar composed the Enigma Variations'

> 'Elgar lived in Malvern'

can be joined by the **connective** 'and' to form a **compound proposition** such as

> 'Elgar composed the Enigma Variations **and** lived in Malvern.'

A **compound proposition** can be described as a proposition made up of two or more simple propositions joined by connectives.

There is a variety of connectives which will now be defined.

Conjunction   AND   ∧

If two propositions are joined by the word AND to form a compound statement, this is called a **conjunction** and is denoted by the symbol ∧.

Example

If **p** is the proposition 'the sun is shining'

and **q** is the proposition 'Jack is wearing sunglasses',

then **p** ∧**q** represents the conjunction 'the sun is shining AND Jack is wearing sunglasses'

> **Note**
>
>   In the printed form, a proposition is shown by a small letter printed bold, eg **a**, **b**.
>
>   When writing by hand it is not easy to make letters bold, so each letter is written with a squiggle underneath it (as with vectors), to indicate that it is a proposition.
>
>   For example, the proposition printed as **p** would be hand written as p̰ .

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Chapter 10 Logic

Disjunction   OR   ∨

If two statements are joined by the word OR to form a compound proposition, this is called a **disjunction** and is denoted by the   symbol ∨.

Example

If **p** is the proposition 'Ann is studying geography'

and **q** is the proposition 'Ann is studying French'

then the disjunction **p** ∨**q** is the compound statement

 'Ann is studying geography OR French.'

The word 'OR' in this context can have two possible meanings.

Can Ann study both subjects?

Think about the meaning of these two sentences.

    'I can deliver your coal on Wednesday or Thursday.'

    'My fire can burn logs or coal.'

The first sentence implies that there is only one delivery of coal and illustrates the **exclusive** use of OR, meaning 'or' but not 'both'. The coal can be delivered on Wednesday or Thursday, but would not be delivered on both days.

The second sentence illustrates the **inclusive** use of 'OR' meaning that the fire can burn either logs or coal, or both logs and coal.

The word 'OR' and the symbol '∨' are used for the **inclusive** OR, which stands for 'and/or'.

The exclusive OR is represented by the symbol ⊕.

Activity 4   Exclusive or inclusive?

Write down three English sentences which use the inclusive OR and three which use the exclusive OR.

Now that a range of connectives is available propositions can be combined into a variety of compound propositions.

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Chapter 10   Logic

Example

Use p, q and r to represent affirmitive (or positive) statements and express the following proposition symbolically.

'Portfolios may include paintings or photographs but not collages.'

Solution

So, let **p** be   'portfolios may include paintings'

    let **q**   be   'portfolios may include photographs'

and let **r**   be   'portfolios may include collages'.

The proposition therefore becomes

$$(p \lor q) \land \sim r.$$

Exercise 10A

1. For each of these compound propositions, use **p**, **q** and **r** to represent affirmative (or positive) statements and then express the proposition symbolically.

   (a) This mountain is high and I am out of breath.

   (b) It was neither wet nor warm yesterday.

   (c) During this school year Ann will study two or three subjects.

   (d) It is not true that $3+7=9$ and $4+4=8$.

2. Let **p** be 'the cooker is working', **q** 'the food supply is adequate' and **r** 'the visitors are hungry'.   Write the following propositions in 'plain English':

   (a) $p \land \sim r$

   (b) $q \land r \land \sim p$

   (c) $r \lor \sim q$

   (d) $\sim r \lor (p \land q)$

   (e) $\sim q \land (\sim p \lor \sim r)$

10.3 Boolean expressions

The system of logic using expressions such as $p \lor q$ and $\sim p \land r$ was developed by the British mathematician   *George Boole* (1815 - 1864).

The laws of reasoning were already well known in his time and Boole was concerned with expressing the laws in terms of a special algebra which makes use of what are known as **Boolean expressions**, such as $\sim a \land b.$

Activity 5   Using plain English

Define three propositions of your own, **p**, **q** and **r**, and write in plain English the meaning of these Boolean expressions.

1. $q \wedge r$ 3. $\sim p \vee (q \wedge r)$

2. $\sim p \wedge r$ 4. $r \wedge (\sim p \vee q)$

Using truth tables

In Section 10.2 a proposition was defined as a statement that is either true or false. In the context of logic, the integers 0 and 1 are used to represent these two states.

     0 represents false

     1 represents true.

Clearly, if a proposition **p** is true then ~**p** is false; also if **p** is false, then $\sim p$ is true.   This can be shown in a **truth table**, as below.

p	$\sim p$
0	1
1	0

The connectives, $\vee$ and $\wedge$ can also be defined by truth tables, as shown below.

p	**q**	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

This truth table shows the truth values (0 or 1) of the conjunction $p \wedge q$.

Since $p \wedge q$ means **p** AND **q**, then $p \wedge q$ can only be true (ie 1) when **p** is true AND **q** is true.

If a negation is used, it is best to add the negation column, eg ~**p**, to the truth table.

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Chapter 10   Logic

Example

Construct the truth table for $p \wedge \sim q$.

Solution

p	q	$\sim q$	$p \wedge \sim q$
0	0	1	0
0	1	0	0
1	0	1	1
1	1	0	0

If **p** AND **~q** are true (ie both are 1) then $p \wedge \sim q$ is true.

Exercise 10B

Construct truth tables for the following.

1. $q \vee r$

2. $\sim p \wedge r$

3. $p \vee \sim r$

4. $\sim p \vee \sim q$

10.4 Compound propositions

More complicated propositions can be represented by truth tables, building up parts of the expression.

Example

Construct the truth table for the compound proposition $(a \vee b) \vee \sim c$.

Solution

a	b	c	$(a \vee b)$	$\sim c$	$(a \vee b) \vee \sim c$
0	0	0	0	1	1
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	1	0	1
1	0	0	1	1	1
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	0	1

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Chapter 10 Logic

Exercise 10C

Construct truth tables for the following:

1. $(a \lor b) \lor c$

2. $a \land (b \land c)$

3. $a \lor (b \lor c)$

4. $(a \land b) \land c$

5. $a \land (b \lor c)$

6. $(a \land b) \lor (a \land c)$

7. $a \lor (b \land c)$

8. $(a \lor b) \land (a \lor c)$

**You should notice that some of your answers in this exercise
are the same. What are the implications of this? Can you
think of similar rules for numbers in ordinary algebra? What
names are given to these properties?**

10.5 What are the implications?

'If I win this race, then I will be in the finals.'

'If the light is red, then you must stop.'

These two sentences show another connective, 'if ... then ...'
which is indicated by the symbol $\Rightarrow$.

$x \Rightarrow y$ is the compound proposition meaning that proposition **x**
implies proposition **y.**

Returning to the compound proposition

'If I win this race, then I will be in the finals',

this can be written as $a \Rightarrow b$ where **a** is the proposition 'I win
the race' and **b** is the proposition 'I will be in the finals'. The
first proposition, **a**, 'I win this race', can be true or false.
Likewise, the second proposition 'I will be in the finals' can be
true or false.

If I win the race (**a** is true) and I am in the final (**b** is true) then
the compound proposition is true ($a \Rightarrow b$ is true).

Activity   6   The implication truth table

If I fail to win the race (**a** is false) and I am not in the final (b is
false), is the compound proposition $a \Rightarrow b$ true or false?

If I win the race but am not in the final (illness, injury), then is
the compound proposition $a \Rightarrow b$ true or false?

By considering these two questions and two others, it is possible
to build up a truth table for the proposition $a \Rightarrow b$. Think about

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Chapter 10   Logic

the other two questions and their answers, and hence complete the
following truth table.

a	**b**	$a \Rightarrow b$
0	0	
0	1	
1	0	
1	1	

The values in this truth table often cause much argument, until it is
realised that the connective $\Rightarrow$ is about implication and not about
cause and effect.

It is not correct to assume that $a \Rightarrow b$ means **a** causes **b** or that **b**
results from **a**.

In fact, the implication connective, $\Rightarrow$, is defined by the values
shown in the truth table whatever the propositions that make up the
compound proposition.

Consider the implication $a \Rightarrow b$

'If it is hot, it is June.'

The only way of being sure that this implication $a \Rightarrow b$ is false is
by finding a time when it is hot but it isn't June; i.e. when **a** is true
but **b** is false. Hence the truth table for $a \Rightarrow b$ is as follows:

a	**b**	$a \Rightarrow b$
0	0	1
0	1	1
1	0	0
1	1	1

In logic, the two propositions which make up a compound
proposition may not be related in the usual sense.

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Chapter 10 Logic

Example

'If Christmas is coming (C), today is Sunday (S).'

C	S	$C \Rightarrow S$
0	0	1
0	1	1
1	0	0
1	1	1

However difficult it may seem to invent a meaning for this implication, the truth table will be exactly the same as before.

Exercise 10D

1. Give the truth values (1 or 0) of these propositions.

    (a) If all multiples of 9 are odd, then multiples of 3 are even.

    (b) If dogs have four legs then cats have four legs.

    (c) If the sea is blue, the sky is green.

    (d) Oxford is in Cornwall if Sheffield is in Yorkshire.

    (e) Pentagons have six sides implies that quadrilaterals have four sides.

2. If **a** represents 'the crops grow', **b** is 'I water the plants' and **c** is 'I spread manure', express these propositions in terms of **a**, **b** and **c**.

    (a) If I water the plants the crops grow.

    (b) I do not spread manure nor do I water the plants and the crops do not grow.

    (c) If I spread manure the crops grow.

    (d) The crops grow if I water the plants and do not spread manure.

    (e) If I do not water the plants, then I spread manure and the crops grow.

3. Using **a**, **b** and **c** from Question 2, interpret the following propositions.

    (a) $(a \wedge b) \vee (a \wedge c)$

    (b) $(c \vee \sim b) \Rightarrow \sim a$

    (c) $a \Rightarrow b \wedge c$

    (d) $\sim a \vee c \Rightarrow b$

10.6 Recognising equivalence

There is a difference between the proposition

'If it is dry, I will paint the door.'

and the proposition

'If, and only if, it is dry, I will paint the door.'

If **p** is 'it is dry' and **q** is 'I will paint the door', then $p \Rightarrow q$ represents the first proposition.

164

The second proposition uses the connective of **equivalence** meaning 'if and only if ' and is represented by the symbol $\Leftrightarrow$, i.e. $p \Leftrightarrow q$ represents the second proposition.

The truth table for $p \Leftrightarrow q$ shown here is more obvious than the truth table for implication.

p	q	$p \Leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

You will see that $p \Leftrightarrow q$ simply means that the two propositions **p** and **q** are true or false together: this accounts for the use of the word 'equivalence'. Note that if you work out the truth table for $q \Leftrightarrow p$ you will get the same results as for $p \Leftrightarrow q$.

Exercise 10E

1. If **a** is a true statement and **b** is false, write down the truth value of:

    (a) $a \Leftrightarrow \sim b$

    (b) $\sim b \Leftrightarrow \sim a$.

2. If **a** is 'the theme park has excellent rides', **b** is 'entrance charges are high' and **c** is 'attendances are large', write in plain English the meaning of:

    (a) $c \Leftrightarrow (a \wedge \sim b)$

    (b) $(\sim c \vee \sim b) \Rightarrow a$.

10.7 Tautologies and contradictions

In the field of logic, a **tautology** is defined as a compound proposition which is **always true** whatever the truth values of the constituent statements.

p	$\sim p$	$p \vee \sim p$
0	1	1
1	0	1

This simple truth table shows that

$p \vee \sim p$ is a tautology.

The opposite of a tautology, called a **contradiction**, is defined as a compound proposition which is **always false** whatever the truth values of the constituent statements.

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Chapter 10 Logic

p	~ p	p∧ ~ p
0	1	0
1	0	0

This simple truth table shows that

$p \wedge \sim p$ is a contradiction.

Example

Is $\left[a \wedge (b \vee \sim b)\right] \Leftrightarrow a$ a tautology or a contradiction?

Solution

The clearest way to find the solution is to draw up a truth table. If the result is always true then the statement is a tautology; if always false then it is a contradiction.

a	b	~ b	$(b \vee \sim b)$	$a \wedge (b \vee \sim b)$	$\left[a \wedge (b \vee \sim b)\right] \Leftrightarrow a$
0	0	1	1	0	1
0	1	0	1	0	1
1	0	1	1	1	1
1	1	0	1	1	1

The truth table shows that, since the compound statement is always true, the example given is a tautology.

Exercise 10F

Decide whether each of the following is a tautology or a contradiction.

1. $(a \Rightarrow b) \Leftrightarrow (a \wedge \sim b)$
2. $\left[a \wedge (a \Rightarrow b)\right] \wedge \sim b$
3. $\sim (a \Rightarrow b) \Rightarrow \left[(b \vee c) \Rightarrow (a \vee c)\right]$

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Chapter 10   Logic

10.8 The validity of an argument

In Section 8.1 the idea of an argument was described as a set of
premises (such as **p**, **q** and **r**) which leads to a conclusion (**c**):

$$\begin{array}{c} \mathbf{p} \\ \mathbf{q} \\ \mathbf{r} \\ \cdot \\ \cdot \\ \cdot \\ \hline \mathbf{c} \end{array}$$

A **valid** argument is one in which, if the premises are true, the
conclusion must be true. An **invalid** argument is one that is not
valid. The validity of an argument can, in fact, be independent
of the truth (or falsehood) of the premises. It is possible to have
a valid argument with a false conclusion or an invalid argument
with a true conclusion. An argument can be shown to be valid if
$p \wedge q \wedge r \wedge .... \Rightarrow c$ is always true (i.e. a tautology).

Example

Represent the following argument symbolically and determine
whether the argument is valid.

If cats are green then I will eat my hat.

I will eat my hat.

Cats are green.

Solution

Write the argument as

$$\begin{array}{c} a \Rightarrow b \\ b \\ \hline \mathbf{a} \end{array}$$

The argument is valid if $(a \Rightarrow b) \wedge b \Rightarrow a$.

a	**b**	$(a \Rightarrow b)$	$(a \Rightarrow b) \wedge b$	$(a \Rightarrow b) \wedge b \Rightarrow a$
0	0	1	0	1
0	1	1	1	0
1	0	0	0	1
1	1	1	1	1

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Chapter 10 Logic

The truth table shows that the argument is not always true (i.e. it is not a tautology) and is therefore invalid. The second line in the truth table shows that the two premises $a \Rightarrow b$ and **b** can both be true with the conclusion **a** being false. In other words, the compound proposition $(a \Rightarrow b) \wedge b \Rightarrow a$ is not always true (i.e. it is not a tautology). Therefore the argument

$$a \Rightarrow b$$
$$\underline{b}$$
$$\mathbf{a}$$

is invalid.

Exercise 10G

Determine whether these arguments are valid.

1.
$$a \Rightarrow b$$
$$\underline{a \Rightarrow c}$$
$$a \Rightarrow (b \wedge c)$$

2.
$$\sim b \Rightarrow \sim a$$
$$\underline{\mathbf{b}}$$
$$\mathbf{a}$$

3.
$$p \Rightarrow q$$
$$\underline{r \Rightarrow \sim q}$$
$$p \Rightarrow \sim r$$

4. Form a symbolic representation of the following argument and determine whether it is valid.

> If I eat well then I get fat.
>
> $\underline{\text{If I don't get rich then I}}$
> $\underline{\text{don't get fat.}}$
>
> I get rich.

10.9 Miscellaneous Exercises

1. Denote the positive (affirmative) statements in the following propositions by **a**, **b**, **c**, ... and express each proposition symbolically.

   (a) Either you have understood this chapter, or you will not be able to do this question.

   (b) 64 and 169 are perfect squares.

   (c) $-4 > -9$ and $4 > -9$.

   (d) This is neither the right time nor the right place for an argument.

   (e) If the wind is blowing from the east, I will go sailing tomorrow.

   (f) The train standing at platform 5 will not leave unless all the doors are shut.

   (g) The telephone rang twice and there was no reply.

   (h) My friend will go to hospital if his back doesn't get better.

2. Draw up a truth table for these propositions:

   (a) $(p \vee \sim q) \Rightarrow q$

   (b) $\left[p \vee (\sim p \vee q)\right] \vee (\sim p \wedge \sim q)$

   (c) $(\sim p \vee \sim q) \Rightarrow (p \wedge \sim q)$

   (d) $\sim p \Leftrightarrow q$

   (e) $(\sim p \wedge q) \vee (r \wedge p)$

   (f) $(p \Leftrightarrow q) \Rightarrow (\sim p \wedge q)$

3. Decide whether each of the following is a tautology:

   (a) $\sim a \Rightarrow (a \Rightarrow b)$

   (b) $\sim (a \vee b) \wedge a$

   (c) $\left[a \wedge (a \Rightarrow b)\right] \Rightarrow a$

   (d) $(a \Rightarrow b) \Leftrightarrow \sim (a \wedge \sim b)$

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Chapter 10   Logic

4. Decide whether each of the following is a contradiction:

   (a) $(a \wedge b) \vee (\sim a \wedge \sim b)$

   (b) $(a \Rightarrow b) \Leftrightarrow (a \wedge \sim b)$

   (c) $\sim (a \wedge b) \vee (a \vee b)$

   (d) $(a \vee b) \Rightarrow \sim (b \vee c)$

5. Formulate these arguments symbolically using **p**, **q** and **r,** and decide whether each is valid.

   (a) If I work hard, then I earn money

   I work hard
   _____
   I earn money

   (b) If I work hard then I earn money

   If I don't earn money then I am not successful
   _____
   I earn money

   (c) I work hard if and only if I am successful

   I am successful
   _____
   I work hard.

   (d) If I work hard or I earn money then I am successful

   I am successful
   _____
   If I don't work hard then I earn money.

*6. *Lewis Carroll* gave many arguments in his book 'Symbolic Logic'. Decide whether the following arguments are valid.

   (a) No misers are unselfish.

   None but misers save egg-shells.
   _____
   No unselfish people save egg-shells.

   (b) His songs never last an hour;

   A song, that lasts an hour, is tedious.
   _____
   His songs are never tedious.

   (c) Babies are illogical;

   Nobody is despised who can manage a crocodile;

   Illogical persons are despised.
   _____
   Babies cannot manage crocodiles.

   (Hint   **a** : persons who are able to manage a crocodile.

   b : persons who are babies

   c : persons who are despised

   d : persons who are logical)

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Chapter 10 Logic