Integration

Section 9: Differential equations

Notes and Examples

These notes contain subsections on

Rates of change

The derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the rate of change of y with respect to x. When time t is used instead of x, then $\frac{\mathrm{d}y}{\mathrm{d}t}$ represents the rate of change of y. For example:

- If x is a displacement, then $\frac{\mathrm{d}x}{\mathrm{d}t}$ is the rate of change of displacement, or the velocity;
- If v is a velocity, then $\frac{\mathrm{d}v}{\mathrm{d}t}$ is the rate of change of velocity, or the acceleration;
- If P represents the number of ants in a colony, then $\frac{\mathrm{d}P}{\mathrm{d}t}$ is the rate of change of population; if this is positive then the population is increasing, if negative then it is declining.
- If M represents the mass of a compound during a chemical process, then $\frac{\mathrm{d}M}{\mathrm{d}t}$ is the rate of change of mass.

Differential equations

Theory often leads us to formulate mathematical models for the rate of change of quantities. For example,

- when a cup of hot water cools, the rate of cooling obviously depends on the temperature;
- when a car accelerates, the acceleration at faster speeds is slower than at slower speeds – so the rate of change of velocity depends on the velocity;
- the rate of population growth depends on size of the population.

In all these cases, we can model the situation by means of a differential equation. This is simply an equation which includes a derivative. If the derivative is of the first order, it is called a first order differential equation.

Formulating differential equations

If a is proportional to b, we can write this as $a \propto b$, and deduce that $a = kb$, where k is a constant, called the constant of proportionality.

Example 1
Formulate a differential equation for the following situations.
(i) The acceleration of a particle is proportional to the square of its velocity.
(ii) The rate of population growth is proportional to the size of the population at any time.
(iii) The rate of cooling of a glass of water is proportional to the difference between its temperature and room temperature.

Solution
(i) Let the velocity be v.

The acceleration is the rate of change of velocity, $\frac{\mathrm{d}v}{\mathrm{d}t}$.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = kv^2 .$$

(ii) Let P be the population size.

The rate of population growth is $\frac{\mathrm{d}P}{\mathrm{d}t}$.

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP .$$

(iii) Let T be the temperature of the water, and T_0 be room temperature.

The rate of change of temperature is $\frac{\mathrm{d}T}{\mathrm{d}t}$, so the rate of cooling is $-\frac{\mathrm{d}T}{\mathrm{d}t}$.

This is proportional to $T - T_0$, so $\frac{\mathrm{d}T}{\mathrm{d}t} = -k(T - T_0)$, where $k > 0$.

Notice the negative sign! This will be crucial to the solution.

Solving differential equations by direct integration

You already know how to solve some differential equations, like the ones in the next example.

Example 2

Solve the differential equations:

(i) $\frac{\mathrm{d}v}{\mathrm{d}t} = 3t$

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{1+x}$, given that when $x = 0$, $y = 1$.

Solution

(i) $\frac{\mathrm{d}v}{\mathrm{d}t} = 3t \Rightarrow v = \int 3t \, \mathrm{d}t$

$$= \tfrac{3}{2}t^2 + c$$

The solution is $v = \tfrac{3}{2}t^2 + c$

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{1+x} \Rightarrow y = \int \frac{2}{1+x} \mathrm{d}x$.

$$= 2\ln|1+x| + c$$

When $x = 0$, $y = 1$ $\Rightarrow 1 = 2\ln 1 + c$

$$\Rightarrow c = 1$$

The solution is $y = 2\ln|1+x| + 1$

A solution like the one in part (i) of Example 2, which involves an arbitrary constant (c), is called the **general solution** of the differential equation. In effect, the solution is a family of functions, one for each value of c.

In part (ii) of Example 2, the solution function must pass through the point $x = 0$, $y = 1$, and this enables you to calculate the value of c. This gives a **particular solution** to the differential equation.

Solving differential equations by separating the variables

Now look at this differential equation:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -3v^2, \text{ and when } t = 0,\ v = 10$$

On the right of this equation is a function of v, not t. To get v, you need to integrate with respect to t, not v.

You solve this problem like this:

Integrate both sides with respect to t

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -3v^2 \Rightarrow \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = -3$$

$$\Rightarrow \int \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t = -\int 3\mathrm{d}t .$$

Now $\int \frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t$ is the same as $\int \frac{1}{v^2}\mathrm{d}v$ or $\int v^{-2}\mathrm{d}v$.

Note that you only need to add a constant of integration to one side of the equation.

$$\Rightarrow \int v^{-2}\mathrm{d}v = -\int 3\mathrm{d}t$$

$$\Rightarrow -v^{-1} = -3t + c\,.$$

$$\Rightarrow \frac{1}{v} = 3t - c$$

When $t = 0$, $v = 10$ $\quad \Rightarrow \frac{1}{10} = -c$

$$\Rightarrow \frac{1}{v} = 3t + \frac{1}{10} = \frac{30t+1}{10}$$

$$\Rightarrow v = \frac{10}{30t+1}\,.$$

In this technique, you arrange for the 'v's on the same side of the equations as the $\mathrm{d}v$, and the 't's on the same side as $\mathrm{d}t$. This is called **separating the variables**. The first few steps can also be written more simply like this:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -3v^2$$

$\Rightarrow \quad \int \frac{1}{v^2}\mathrm{d}v = \int -3\mathrm{d}t$, and so on…

Here are some more examples.

Example 3

The rate of growth of a population after t months P of ants is proportional to the number of ants. At time $t = 0$, there are 500 ants, and after 1 month the population has risen to 750. Find the population after 3 months.

Solution

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\text{, where } k \text{ is a constant.}$$

Remember that the inverse of the natural logarithm function is the exponential function

$$\Rightarrow \quad \int \frac{1}{P}\mathrm{d}P = \int k\mathrm{d}t$$

$$\Rightarrow \quad \ln P = kt + c$$

$$\Rightarrow \quad P = \mathrm{e}^{kt+c}$$

$$= \mathrm{e}^{kt} \times \mathrm{e}^{c}$$

$$= A\mathrm{e}^{kt}$$

When dealing with exponentials, it is usual to simplify the solution by replacing the constant e^c by a new constant A.

When $t = 0$, $P = 500$:

$$500 = A\mathrm{e}^{k0} = A$$

$$\Rightarrow \quad P = 500\mathrm{e}^{kt}$$

When $t = 1$, $P = 750$:

$$750 = 500\mathrm{e}^{k}$$

$$\Rightarrow \quad \mathrm{e}^{k} = 1.5$$

$$\Rightarrow \quad P = 500\left(e^{k}\right)^{t}$$
$$= 500 \times 1.5^{t}$$

When $t = 3$, $P = 500 \times 1.5^{3} = 1687.5$

Example 4

Find the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x(1+x)}$.

Solution

$$\frac{dy}{dx} = \frac{y}{x(1+x)}$$

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int \frac{1}{x(1+x)}\,dx$$

By partial fractions,

The details of finding the partial fractions are omitted here.

$$\Rightarrow \quad \int \frac{1}{y}\,dy = \int \left(\frac{1}{x} - \frac{1}{1+x}\right) dx$$

$$\Rightarrow \quad \ln|y| = \ln|x| - \ln|1+x| + c$$
$$= \ln|x| - \ln|1+x| + \ln A$$
$$= \ln\left|\frac{Ax}{1+x}\right|$$

Replace c with a new constant $\ln A$ to give a simpler expression

$$\Rightarrow \quad y = \frac{Ax}{1+x}$$

You can omit the modulus signs at this stage – the value of A will deal with this.

You can practice solving first order differential equations with separable variables using the interactive test **Differential equations: first order** (all the questions are of the same type).