AS PURE MATHS REVISION NOTES

1 SURDS

- A root such as $\sqrt{3}$ that cannot be written exactly as a fraction is IRRATIONAL
- An expression that involves irrational roots is in SURD FORM e.g. $2\sqrt{3}$
- $3 + \sqrt{2}$ and $3 - \sqrt{2}$ are CONJUGATE/COMPLEMENTARY surds – needed to rationalise the denominator

SIMPLIFYING $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Simplify $\sqrt{75} - \sqrt{12}$

$= \sqrt{5 \times 5 \times 3} - \sqrt{2 \times 2 \times 3}$

$= 5\sqrt{3} - 2\sqrt{3}$

$= 3\sqrt{3}$

RATIONALISING THE DENOMINATOR (removing the surd in the denominator)

$a + \sqrt{b}$ and $a - \sqrt{b}$ are CONJUGATE/COMPLEMENTARY surds – the product is always a rational number

Rationalise the denominator $\frac{2}{2-\sqrt{3}}$

$$= \frac{2}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{4+2\sqrt{3}}{4+2\sqrt{3}-2\sqrt{3}-3}$$

$= 4 + 2\sqrt{3}$

Multiply the numerator and denominator by the conjugate of the denominator

2 INDICES

Rules to learn

$x^a \times x^b = x^{a+b}$ $x^{-a} = \frac{1}{x^a}$ $x^0 = 1$

$x^a \div x^b = x^{a-b}$ $x^{\frac{1}{n}} = \sqrt[n]{x}$

$(x^a)^b = x^{ab}$ $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Solve the equation

$25^x = (5^2)^x$

$$3^{2x} \times 25^x = 15$$

$$(3 \times 5)^{2x} = (15)^1$$

$$2x = 1$$

$$x = \frac{1}{2}$$

Simplify

$(x-y)^{\frac{3}{2}}$
$= (x-y)^{\frac{1}{2}}(x-y)$

$$2x(x-y)^{\frac{3}{2}} + 3(x-y)^{\frac{1}{2}}$$

$$(x-y)^{\frac{1}{2}}(2x(x-y)+3)$$

$$(x-y)^{\frac{1}{2}}(2x^2-2xy+3)$$

3 QUADRATIC EQUATIONS AND GRAPHS

Factorising identifying the roots of the equation $ax^2 + bc + c = 0$

- Look out for the difference of 2 squares $x^2 - a^2 = (x - a)(x + a)$
- Look out for the perfect square $x^2 + 2ax + a^2 = (x + a)^2$ or $x^2 - 2ax + a^2 = (x - a)^2$
- Look out for equations which can be transformed into quadratic equations

Solve $x + 1 - \frac{12}{x} = 0$

$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x = 3,\ x = -4$$

Solve $6x^4 - 7x^2 + 2 = 0$

Let $z = x^2$

$$6z^2 - 7z + 2 = 0$$
$$(2z - 1)(3z - 2) = 0$$
$$z = \frac{1}{2} \quad z = \frac{2}{3}$$
$$x = \pm\sqrt{\frac{1}{2}} \quad x = \pm\sqrt{\frac{2}{3}}$$

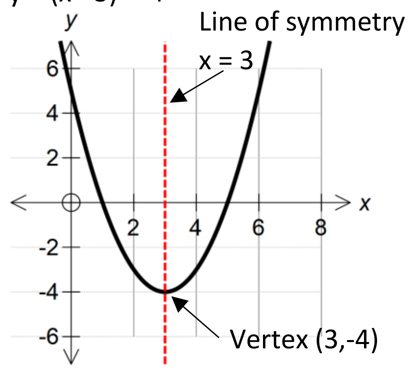
Completing the square - Identifying the vertex and line of symmetry

In completed square form

$y = (x + a)^2 - b$ the vertex is $(-a, b)$

the equation of the line of symmetry is $x = -a$

$y = (x - 3)^2 - 4$



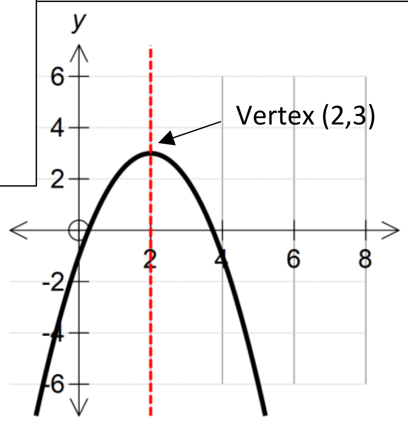
Sketch the graph of

$y = 4x - x^2 - 1$

$y = -(x^2 - 4x) - 1$

$y = -((x - 2)^2 - 4) - 1$

$y = -(x-2)^2 + 3$



Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ for solving $ax^2 + bx + c = 0$

The **DISCRIMINANT $b^2 - 4ac$** can be used to identify the number of solutions

$b^2 - 4ac > 0$ there are 2 real and distinct roots (the graphs crosses the x- axis in 2 places)

$b^2 - 4ac = 0$ the is a single repeated root (the x-axis is a tangent to the graph)

$b^2 - 4ac < 0$ there are no 2 real roots (the graph does not touch or cross the x-axis)

4 SIMULTANEOUS EQUATIONS

Solving by elimination

$3x - 2y = 19$	$\times 3$	$9x - \mathbf{6y} = 57$			
$2x - 3y = 21$	$\times 2$	$\underline{4x - \mathbf{6y} = 42}$			
		$5x - 0y = 15$	$x = 3$	$(9 - 2y = 19)$	$y = -5$

Solving by substitution

$x + y = 1$ rearranges to $y = 1 - x$

$x^2 + y^2 = 25$

$x^2 + (1 - x)^2 = 25$

$x^2 + 1 - 2x + x^2 - 25 = 0$

$2x^2 - 2x - 24 = 0$

$2(x^2 - x - 12) = 0$

$2(x - 4)(x + 3) = 0$ $\quad x = 4 \quad x = -3$

$\quad y = -3 \quad y = 4$

If when solving a pair of simultaneous equations, you arrive with a quadratic equation to solve, this can be used to determine the relationship between the graphs of the original equations

Using the discriminant

$b^2 - 4ac > 0$ the graphs intersect at 2 distinct points

$b^2 - 4ac = 0$ the graphs intersect at 1 point (tangent)

$b^2 - 4a < 0$ the graphs do not intersect

5 INEQUALITIES

Linear Inequality

This can be solved like a linear equation except that

Multiplying or Dividing by a negative value reverses the inequality

Solve $10 - 3x < 4$

$-3x < -6$

$x > 2$

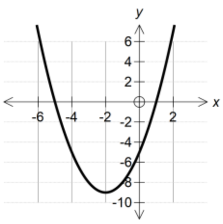
Quadratic Inequality – always a good idea to sketch the graph!

Solve $\mathbf{x^2 + 4x - 5 < 0}$

$x^2 + 4x - 5 = 0$

$(x - 1)(x + 5) = 0$

$x = 1 \quad x = -5$



$x^2 + 4x - 5 < 0$

$-5 < x < 1$

which can be written as

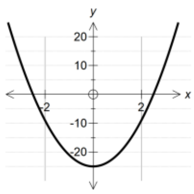
$\{x : x > -5\} \cap \{x : x < 1\}$

Solve $\mathbf{4x^2 - 25 \geq 0}$

$4x^2 - 25 = 0$

$(2x - 5)(2x + 5) = 0$

$x = \frac{5}{2} \quad x = -\frac{5}{2}$



$4x^2 - 25 \geq 0$

$x \leq -\frac{5}{2}$ or $x \geq \frac{5}{2}$

which can be written as

$\{x : x \leq -\frac{5}{2}\} \cup \{x : x \geq \frac{5}{2}\}$

6 GRAPHS OF LINEAR FUNCTIONS

$$y = mx + c$$

the line intercepts the y axis at (0, c)

Gradient = $\frac{change\ in\ y}{change\ in\ x}$

Positive gradient

Negative gradient

Finding the equation of a line with gradient m through point (a,b)

Use the equation (y – b) = m(x – a)

If necessary rearrange to the required form (ax + by = c or y = mx – c)

Parallel and Perpendicular Lines

$y = m_1x + c_1$ $y = m_2x + c_2$

If $\mathbf{m_1 = m_2}$ then the lines are **PARALLEL**

If $\mathbf{m_1 \times m_2 = -1}$ then the lines are **PERPENDICULAR**

> Find the equation of the line perpendicular to the line y – 2x = 7 passing through point (4, -6)
>
> Gradient of y – 2x = 7 is 2 (y = 2x + 7)
> Gradient of the perpendicular line = - ½ (2 × -½ = -1)
>
> Equation of the line with gradient –½ passing through (4, -6)
> (y + 6) = -½(x – 4)
> 2y + 12 = 4 – x
> x + 2y = - 8

Finding mid-point of the line segment joining (a,b) and (c,d)

Mid-point = $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

Calculating the length of a line segment joining (a,b) and (c,d)

Length = $\sqrt{(c-a)^2 + (d-b)^2}$

7 CIRCLES

A circle with centre (0,0) and radius r has the equations $x^2 + y^2 = r^2$

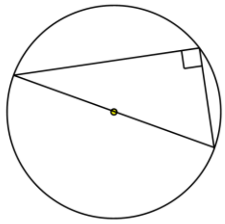
A circle with centre (a,b) and radius r is given by $(x - a)^2 + (y - b)^2 = r^2$

Finding the centre and the radius (completing the square for x and y)

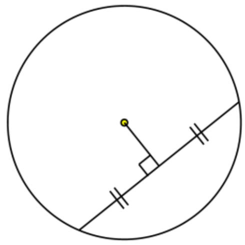
> Find the centre and radius of the circle $x^2 + y^2 + 2x – 4y – 4 = 0$
> $x^2 + 2x + y^2 – 4y – 4 = 0$
> $(x + 1)^2 – 1 + (y – 2)^2 – 4 – 4 = 0$
> $(x + 1)^2 + (y – 2)^2 = 3^2$
> Centre (-1, 2) Radius = 3

The following circle properties might be useful

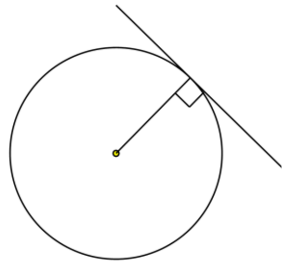
Angle in a semi-circle is a right angle



The perpendicular from the centre to a chord bisects the chord



The tangent to a circle is perpendicular to the radius



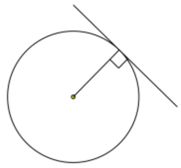
Finding the equation of a tangent to a circle at point (a,b)

The gradient of the tangent at (a,b) is perpendicular to the gradient of the radius which meets the circumference at (a, b)

Find equation of the tangent to the circle $x^2 + y^2 - 2x - 2y - 23 = 0$ at the point (5,4)

$(x - 1)^2 + (y - 1)^2 - 25 = 0$

Centre of the circle (1,1)



Gradient of radius $= \frac{4-1}{5-1} = \frac{3}{4}$ Gradient of tangent $= -\frac{4}{3}$

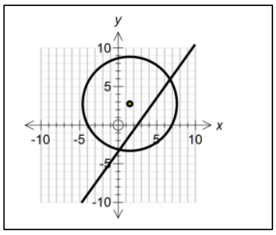
Equation of the tangent $(y - 4) = -\frac{4}{3}(x - 5)$ $3y - 12 = 20 - 4x$

$4x + 3y = 32$

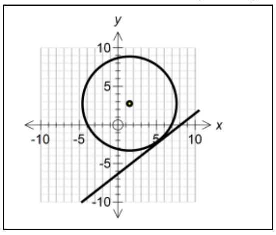
Lines and circles Solving simultaneously to investigate the relationship between a line and a circle will result in a quadratic equation.

Use the discriminant to determine the relationship between the line and the circle

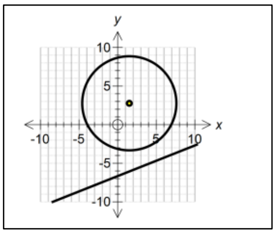
$b^2 - 4ac > 0$



$b^2 - 4ac = 0$ (tangent)



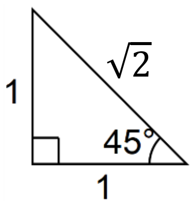
$b^2 - 4ac < 0$



8 TRIGONOMETRY

You need to learn ALL of the following

Exact Values



$\sin 45° = \frac{\sqrt{2}}{2}$

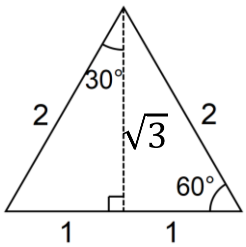
$\cos 45° = \frac{\sqrt{2}}{2}$

$\tan 45° = 1$

$\sin 30° = \frac{1}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$

$\tan 30° = \frac{1}{\sqrt{3}}$



$\sin 60° = \frac{\sqrt{3}}{2}$

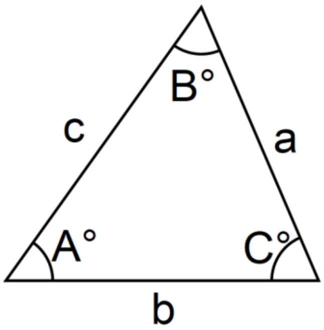
$\cos 60° = \frac{1}{2}$

$\tan 60° = \sqrt{3}$

Cosine Rule $a^2 = b^2 + c^2 - 2bc \cos A$

Sine Rule $\dfrac{a}{Sin\ A} = \dfrac{b}{Sin\ B} = \dfrac{c}{Sin\ C}$

Area of a triangle $\frac{1}{2}abSinC$

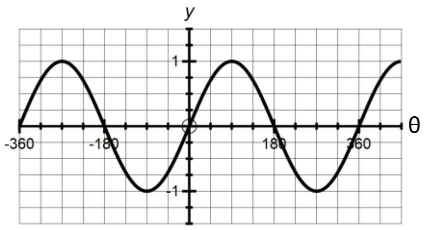


Identities

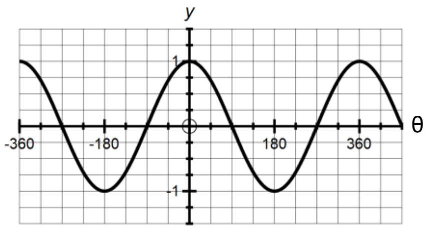
$\sin^2 x + \cos^2 x = 1$ $\tan x = \dfrac{\sin x}{\cos x}$

Graphs of Trigonometric Functions

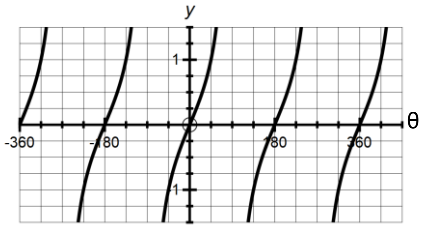
y = sin θ



y = cos θ



y =tan θ



Solve the equation $\sin^2 2\theta + \cos 2\theta + 1 = 0$ $0° \le \theta \le 360°$

$(1 - \cos^2 2\theta) + \cos 2\theta + 1 = 0$

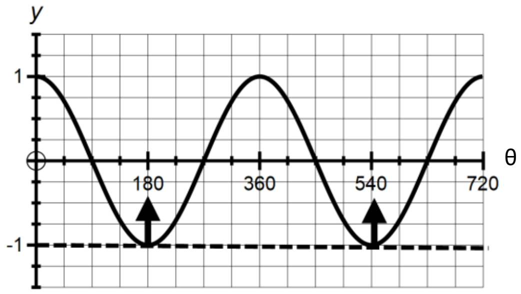
$\cos^2 2\theta - \cos 2\theta - 2 = 0$

$(\cos 2\theta - 2)(\cos 2\theta + 1) = 0$

$\cos 2\theta = 2$ (no solutions)

$\cos 2\theta = -1$ $2\theta = 180°, 540°$

$\theta = 90°, 270°$



9 POLYNOMIALS

- A polynomial is an expression which can be written in the form $ax^n + bx^{n-1} + cx^{n-2}$ ... ... when a,b, c are constants and n is a positive integer.
- The **ORDER** of the polynomial is the highest power of x in the polynomial

Algebraic Division

Polynomials can be divided to give a **Quotient** and **Remainder**

Divide $x^3 - x^2 + x + 15$ by $x + 2$

$$
\begin{array}{rrrrr}
 & x^2 & -3x & +7 & \leftarrow \textit{Quotient} \\
\cline{2-4}
x \quad +2 \;\big) & x^3 \quad -x^2 & +x & +15 & \\
 & x^3 \quad +2x^2 & & & \\
\cline{2-2}
 & -3x^2 & +x & & \\
 & -3x^2 & -6x & & \\
\cline{2-3}
 & & 7x & +15 & \\
 & & 7x & +14 & \\
\cline{3-4}
 & & & 1 & \leftarrow \textit{Remainder}
\end{array}
$$

Factor Theorem

The factor theorem states that if (x – a) is a factor of f(x) then f(a) = 0

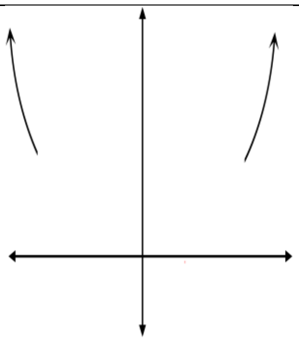
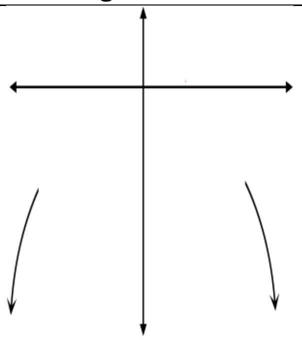
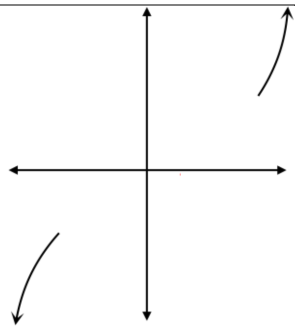
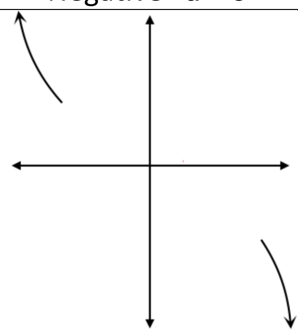
Show that (x – 3) is a factor of x^3 -19x + 30 = 0
f(x) = x^3 – 19x + 30
f(3) = 3^3 - 19×3 + 30
= 0
f(3) = 0 so (x – 3) is a factor

Sketching graphs of polynomial functions

To sketch a polynomial

- Identify where the graph crosses the y-axis (x = 0)
- Identify the where the graph crosses the x-axis, the roots of the equation y = 0
- Identify the general shape by considering the **ORDER** of the polynomial

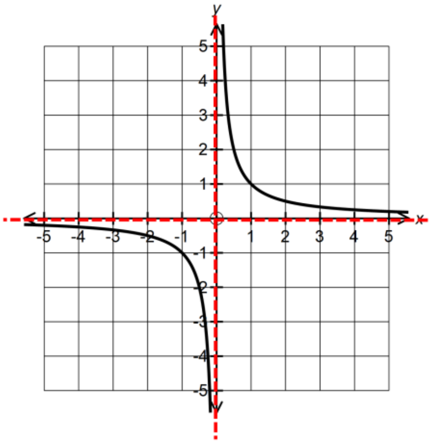
y = $ax^n + bx^{n-1} + cx^{n-2}$ … …

n is even		n is odd	
Positive **a > 0**	Negative **a < 0**	Positive **a > 0**	Negative **a < 0**
			

10 GRAPHS AND TRANSFORMATIONS

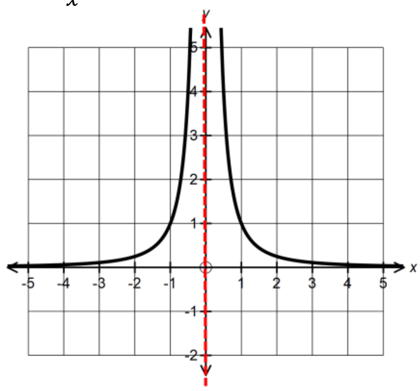
3 graphs to recognise

$y = \frac{1}{x}$



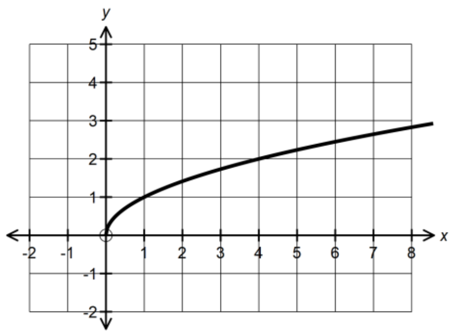
Asymptotes x= 0 and y = 0

$y = \frac{1}{x^2}$



Asymptote x = 0

$y = \sqrt{x}$



TRANSLATION

To find the equation of a graph after a translation of $\begin{bmatrix} a \\ b \end{bmatrix}$ replace x with (x - a) and replace y with (y – b)

In function notation
y = f(x) is transformed to y = f(x -a) + b

The graph of $y = x^2 - 1$ is translated by vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Write down the equation of the new graph

$(y + 2) = (x - 3)^2 - 1$

$y = x^2 - 6x + 6$

REFLECTION

To reflect in the x-axis replace y with -y (y = -f(x))

To reflect in the y- axis replace x with -x (y = f(-x))

STRETCHING

To stretch with scale factor k in the x direction (parallel to the x-axis) replace x with $\frac{1}{k}x$ $y = f(\frac{1}{k}x)$

To stretch with scale factor k in the y direction (parallel to the y-axis) replace y with $\frac{1}{k}y$ $y = kf(x)$

Describe a stretch that will transform $y = x^2 + x - 1$ to the graph $y = 4x^2 + 2x - 1$

$y = (2x)^2 + (2x) - 1$

x has been replaced by 2x which is a **stretch of scale factor ½ parallel to the x-axis**

11 BINOMIAL EXPANSIONS

Permutations and Combinations

- The number of ways of arranging n distinct objects in a line is n! = n(n - 1)(n - 2)….3 × 2 × 1
- The number of ways of arranging a selection of r object from n is ${}_nP_r = \frac{n!}{(n-r)!}$
- The number of ways of picking r objects from n is ${}_nC_r = \frac{n!}{r!(n-r)!}$

A committee comprising of 3 males and 3 females is to be selected from a group of 5 male and 7 female members of a club. How many different selections are possible?

Female Selection ${}_7C_3 = \frac{7!}{3!4!} = 35$ ways

Male Selection ${}_5C_3 = \frac{5!}{3!2!} = 10$ ways

Total number of different selections = 35 × 10 = 350

Expansion of $(1+x)^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2}x^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}x^3 \ldots\ldots\ldots\ldots + nx^{n-1} + x^n$$

Use the binomial expansion to write down the first four terms of $(1 - 2x)^8$

$$(1-2x)^8 = 1 + 8 \times (-2x) + \frac{8 \times 7}{1 \times 2}(-2x)^2 + \frac{8 \times 7 \times 6}{1 \times 2 \times 3}(-2x)^3$$

$$= 1 - 16x + 112x^2 - 448x^3$$

Expansion of $(a+b)^n$

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3}a^{n-3}b^3 \ldots\ldots\ldots\ldots + nab^{n-1} + b^n$$

Find the coefficient of the x^3 term in the expansion of $(2 + 3x)^9$

$(3x)^3$ must have 2^6 as part of the coefficient (3 + 6 = 9)

$\frac{9\times8\times7}{1\times2\times3} \times 2^6 \times (3x)^3 = 145152\ (x^3)$

12 DIFFERENTIATION

- The gradient is denoted by $\frac{dy}{dx}$ if y is given as a function of x
- The gradient is denoted by f'(x) is the function is given as f(x)

$y = x^n \quad \frac{dy}{dx} = nx^{n-1}$ $\qquad y = ax^n \quad \frac{dy}{dx} = nax^{n-1}$ $\qquad y = a \quad \frac{dy}{dx} = 0$

<u>Using Differentiation</u>

Tangents and Normals

The gradient of a curve at a given point = gradient of the tangent to the curve at that point
The gradient of the **normal** is perpendicular to the gradient of the tangent that point

Find the equation of the normal to the curve y = 8x – x² at the point (2,12)

$\frac{dy}{dx} = 8 - 2x$ Gradient of tangent at (2,12) = 8 – 4 = 4

Gradient of the normal = - ¼ (y - 12) = -¼ (x – 2)

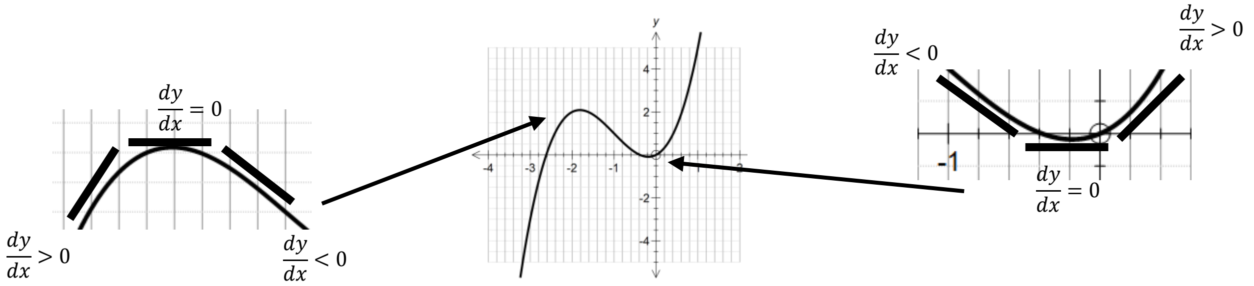
4y + x = 50

Stationary (Turning) Points

- The points where $\frac{dy}{dx} = 0$ are stationary points (turning points) of a graph
- The nature of the turning points can be found by:

Calculating the gradient close to the point

Maximum point **Minimum Point**



Differentiating (again) to find $\frac{d^2y}{dx^2}$ or f''(x)

Maximum if $\frac{d^2y}{dx^2} < 0$

Minimum if $\frac{d^2y}{dx^2} > 0$

Find and determine the nature of the turning points of the curve y = $2x^3 - 3x^2 + 18$

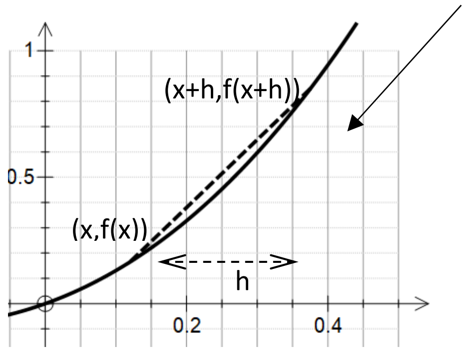
$\frac{dy}{dx} = 6x^2 - 6x$ $\frac{dy}{dx} = 0$ at a turning point

6x(x - 1) = 0 Turning points at (0, 18) and (1,17)

$\frac{d^2y}{dx^2}$ = 12x – 6 x = 0 $\frac{d^2y}{dx^2} < 0$ (0,18) is a maximum

x = 1 $\frac{d^2y}{dx^2} > 0$ (1,17) is a minimum

Differentiation from first principles



As h approaches zero the gradient of the chord gets closer to being the gradient of the tangent at the point

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

Find from first principles the derivative of $x^3 - 2x + 3$

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x+h)^3-2(x+h)+3-(x^3-2x+3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^3+3x^2h+3xh^2+h^3-2x-2h+3-x^3+2x-3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{3x^2h+3xh^2+h^3-2h}{h}\right)$$

$$= \lim_{h\to 0}(3x^2 + 3xh + h^2 - 2)$$

$$= 3x^2 - 2$$

13 INTEGRATION

Integration is the reverse of differentiation

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$ (c is the constant of integration)

Given that f'(x) = $8x^3 - 6x$ and that f(2) = 9, find f(x)

f(x) = $\int 8x^3 - 6x\,dx = 2x^4 - 3x^2 + c$

f(2) = 9 $2\times2^4 - 3\times2^2 + c = 9$

20 + c = 9

c = -11

f(x) = $2x^4 - 3x^2 - 11$

AREA UNDER A GRAPH

The area under the graph of y = f(x) bounded by x = a, x = b and the x-axis is found by evaluating the **definite integral** $\int_a^b f(x)dx$

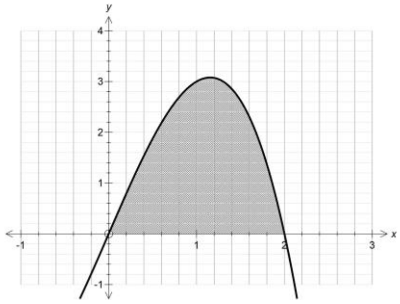
Calculate the area under the graph y = 4x – x³ between x = 0 and x = 2

$\int_0^2 4x - x^3 dx$

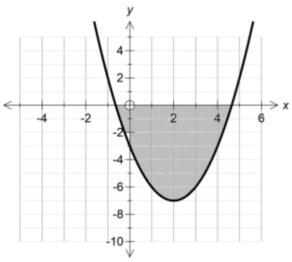
$= \left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= (8-4) - (0-0)$

$= 4$



An area below the x-axis has a **negative value**



14 VECTORS

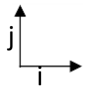
A vector has two properties **magnitude** (size) and **direction**

NOTATION

Vectors can be written as

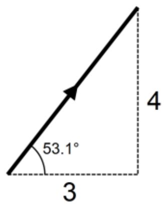
a = $\binom{3}{4}$

a = 3i + 4j where i and j perpendicular vectors both with magnitude 1

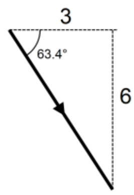


Magnitude-direction form (5, 53.1°) also known as **polar** form

The direction is the angle the vector makes with the **positive x axis**



Express the vector **p** = 3i – 6j in polar form



|**p**| = $\sqrt{3^2 + (-6)^2}$

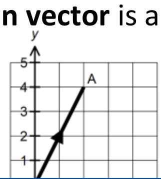
= $3\sqrt{5}$

p = ($3\sqrt{5}$, 63.4°)

The **Magnitude** of vector **a** is denoted by |**a**| and can be found using Pythagoras |**a**| = $\sqrt{3^2 + 4^2}$

A **Unit Vector** is a vector which has magnitude 1

A **position vector** is a vector that starts at the origin (it has a fixed position)



$\overrightarrow{OA} = \binom{2}{4}$ $2i + 4j$

ARITHMETIC WITH VECTORS

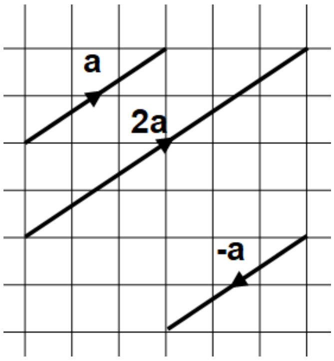
Multiplying by a scalar (number)

$\mathbf{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ 3i + 2j

$2\mathbf{a} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$ 6i + 4j

a and **2a** are **parallel** vectors

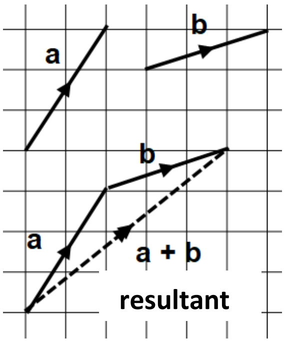
Multiplying by -1 reverses the direction of the vector



Addition of vectors

$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

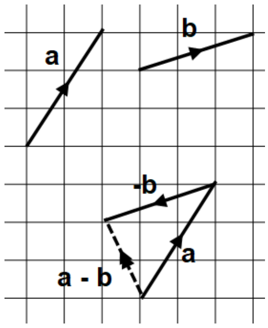


Subtraction of vectors

$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

This is really **a + -b**



A and B have the coordinates (1,5) and (-2,4).

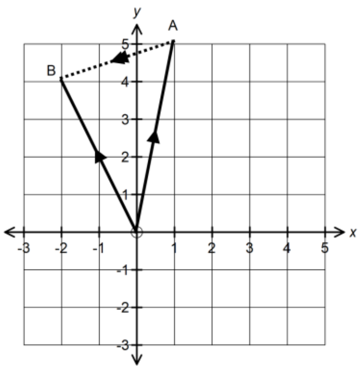
a) Write down the position vectors of A and B

$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$ $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$

b) Write down the vector of the line segment joining A to B

$$\overrightarrow{\boldsymbol{AB}} = -\overrightarrow{\boldsymbol{OA}} + \overrightarrow{\boldsymbol{OB}} \quad \boldsymbol{or} \quad \overrightarrow{\boldsymbol{OB}} - \overrightarrow{\boldsymbol{OA}}$$

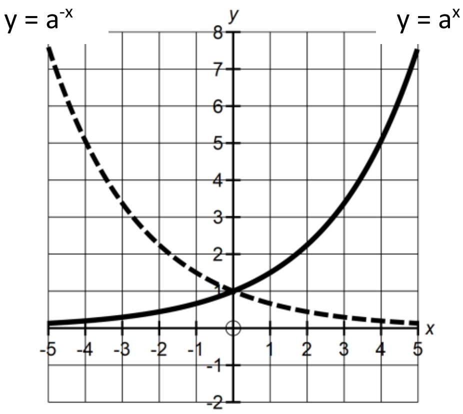
$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$



15 LOGARITHMS AND EXPONENTIALS

- A function of the form $y = a^x$ is an exponential function
- The graph of $y = a^x$ is positive for all values of x and passes through (0,1)
- A logarithm is the inverse of an exponential function

$y = a^x$ $x = \log_a y$



Logarithms – rules to learn

$\log_a a = 1$ $\log_a 1 = 0$ $\log_a a^x = x$ $a^{\log x} = x$

$\log_a m + \log_a n = \log_a mn$ $\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$ $k\log_a m = \log_a m^k$

Write the following in the form alog 2 where a is an integer 3log 2 + 2log 4 – ½log16

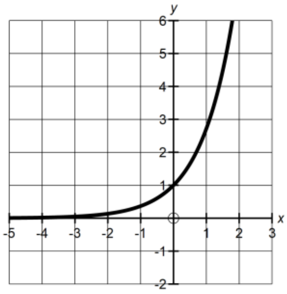
Method 1 : log 8 + log 16 – log 4 = log $\left(\frac{8\times16}{4}\right)$ = log 32 = 5log 2

Method 2 : 3log 2 + 4log 2 – 2log 2 = 5log 2

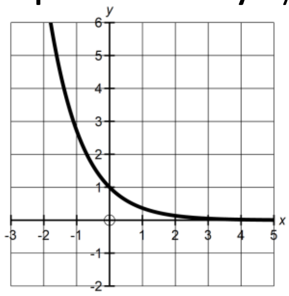
An equation of the form $a^x = b$ can be solved by taking logs of both sides

The exponential function $y = e^x$

Exponential Growth $y = e^x$



Exponential Decay $y = e^{-x}$



The inverse of $y = e^x$ is the **natural logarithm** denoted by **ln x**

Solve $2e^{x-2} = 6$ leaving your answer in exact form

$$e^{x-2} = 3$$
$$\ln(e^{x-2}) = \ln 3$$
$$x - 2 = \ln 3$$
$$x = \ln 3 + 2$$

The rate of growth/decay to find the 'rate of change' you need to differentiate to find the gradient

LEARN THIS

$$y = Ae^{kx} \qquad \frac{dy}{dx} = Ake^{kx}$$

The number of bacteria P in a culture is modelled by
$P = 600 + 5e^{0.2t}$ where t is the time in hours from the start of the experiment. Calculate the rate of growth after 5 hours

$P = 600 + 15e^{0.2t}$ $\frac{dP}{dt} = 3e^{0.2t}$

t = 5 $\frac{dP}{dt} = 3e^{0.2\times5}$

= 8.2 bacteria per hour

MODELLING CURVES

Exponential relationships can be changed to a linear form y = mx + c allowing the constants m and c to be 'estimated' from a graph of plotted data

$y = Ax^n$ $\quad \log y = \log (Ax^n) \quad \log y = n \log x + \log A$

$\quad y \quad = mx \quad + \; c$

Plot log y against log x. n is the gradient of the line and log A is the y axis intercept

$y = Ab^x$ $\quad \log y = \log (Ab^x) \quad \log y = x \log b + \log A$

$\quad y \quad = mx \quad + \; c$

Plot log y against x. log b is the gradient of the line and log A is the y axis intercept

V and x are connected by the equation **$V = ax^b$**

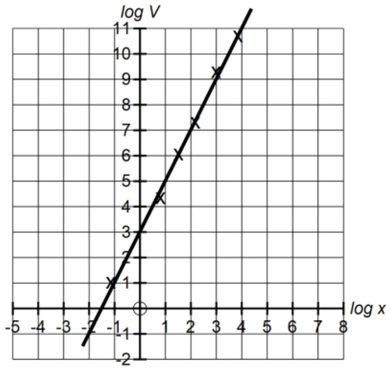
The equation is reduced to linear form by taking logs

log V = b log x + log a

(y = mx + c) (log V plotted against log x)

From the graph b = 2

log a = 3 $\quad a = 10^3$



Gradient = 2

Intercept = 3

16 PROOF

Notation *If* $x = 3$ *then* $x^2 = 9$

$\Rightarrow$ $\quad x = 3 \Rightarrow x^2 = 9$

x = 3 is a condition for $x^2 = 9$

$\Leftarrow$ $\quad x = 3 \Leftarrow x^2 = 9$ is **not true** as x could = - 3

$\Leftrightarrow$ $\quad x + 1 = 3 \Leftrightarrow x = 2$

Useful expressions 2n (an even number) 2n + 1 (an odd number)

Prove that the difference between the squares of any consecutive even numbers is a multiple of 4

Consecutive even numbers 2n, 2n + 2

$(2n + 2)^2 - (2n)^2$

$4n^2 + 8n + 4 - 4n^2$

$=8n + 4$

$=4(2n +1)$ a multiple of 4

Find a **counter example** for the statement '2n + 4 is a multiple of 4'

n = 2 4 + 4 = 8 a multiple of 4

n = 3 6 + 4 = 10 NOT a multiple of 4